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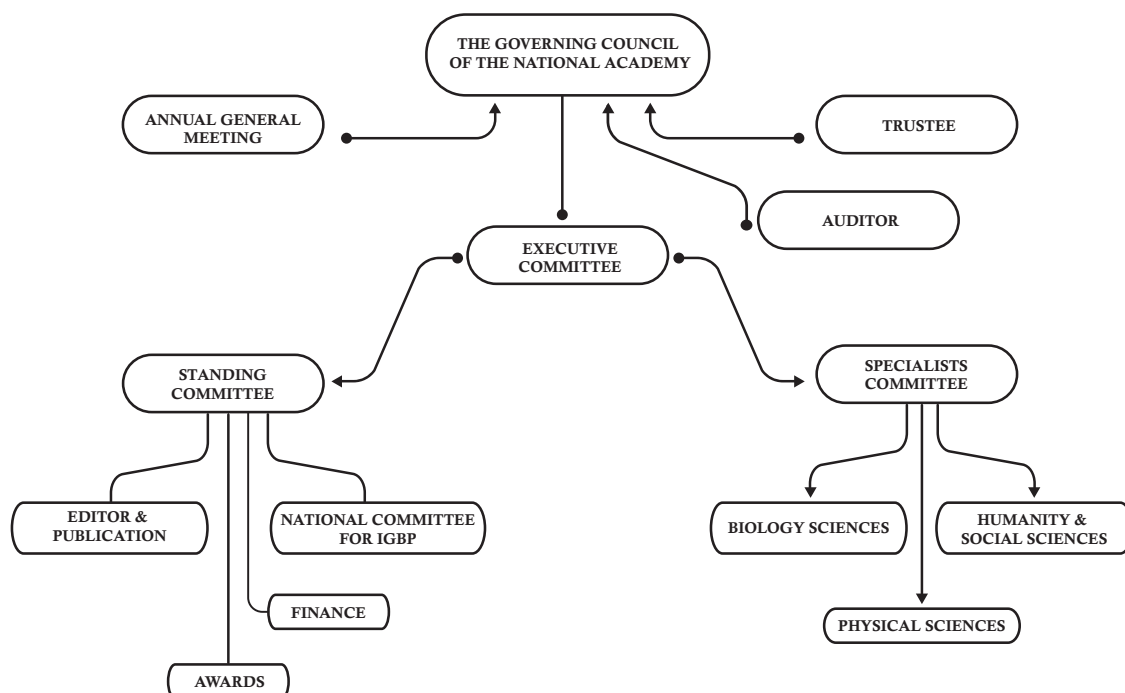
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APPLICATION OF NON-BIRTH-DEATH CONTINUOUS TIME MARKOV CHAIN AVAILABILITY MODELS WITH EXPONENTIALLY AND NON-EXPONENTIALLY DISTRIBUTED UNPLANNED OUTAGES TO CISCO 12000 GSR

Musiga L. A., Ottieno J. A. M. and Weke P.G. O.

School of Mathematics, University of Nairobi, Kenya

Emails: lmusiga@uonbi.ac.ke, joseph.otieno@uonbi.ac.ke, pweke@uonbi.ac.ke

ABSTRACT

Mechanical and electronic machines usually undergo failures and repairs leading to interruptions in service provision. Both the manufacturers and the machine users attach paramount importance to the availability of the systems. The failures that the machines undergo are not nearest neighbour state only, akin to a non-birth-death process. In this paper the non-birth-death continuous time Markov chain availability models with exponentially and non-exponentially distributed unplanned outages are applied to Cisco 12000 GSR, an engineering system that undergoes exponentially and non-exponentially distributed failures. The study found that the availability of the system is 0.999342435, which compares favourably to the system steady-state availability of 0.99978681.

Keywords: Non-birth-death Continuous Time Markov Chain (CTMC), Availability, Cisco 12000 GSR, Exponentially and Hypoexponentially Distributed Unplanned Outages.

1 INTRODUCTION

The advent of the industrial revolution led to both mechanical and electronic devices ensuring accurate mass production. Unfortunately, machines occasionally undergo failures which lead to stoppage of service provision and hence necessitate repair services. The failures are not restricted to the nearest neighbour state thus follow a non-birth-death process, Trivedi (2001). Of paramount importance to the manufacturers and users of machines is the degree of service provision namely, availability. Previous research works have been conducted on systems availability but not by considering the distributions

followed by the failures. In this work, the non-birth-death CTMC availability models with exponentially and non-exponentially distributed unplanned outages are applied to Cisco 12000 GSR.

This paper is organized as follows. In section 2 the previous works on system availability are presented and the CTMC availability models with exponentially and non-exponentially distributed unplanned outages are presented in Section 3. Section 4 is the system description of Cisco 12000 GSR upon which the non-birth-death CTMC availability models are applied. Section 5 comprises the CTMC models of the modules of Cisco 12000 GSR while the input parameter values of Cisco 12000 GSR are displayed in Section 6. The non-birth-death CTMC availability models with exponentially and non-exponentially distributed unplanned outages are applied to Cisco 12000 GSR in Section 7. The paper concludes with discussions in Section 8.

2 RELATED WORKS

Chiang and Chen (2007) proposed a new multi-objective genetic algorithm for the resolution of the availability allocation and optimization problems of a repairable parallel-series system. On comparing the proposed multi-objective genetic algorithm to three other multi-objective genetic algorithms, the proposed multi-objective genetic algorithm provided higher solution quality and greater efficiency in computations.

Smith *et al.* (2008) combined a high-level fault tree model with lower level Markov models. They determined the steady-state availabilities for stand-alone and clustered blade servers. Maciejewski and Caban (2008) proposed an efficient Beta distribution based approach to the estimation of the availability of a repairable system within a fixed time horizon. The approach was found to have great practical impact.

Angus *et al.* (2012) considered age replacement maintenance policy for availability whereby the system was restored whenever it failed or the system aged without failure to a preventive maintenance stage. The restoration period was random, dependant on its' being precipitated by a failure or by a preventive maintenance action. The optimal preventive

maintenance rate was considered, subject to realistic conditions. The results obtained were useful to systems engineers for critical design decisions.

Liu (2013) combined three methods, namely, the simulated annealing method, tabu search method and non-equilibrium simulated annealing method with a genetic algorithm for the solution of the system availability optimization problem. The optimal maintenance rate, reliability level and allocation of component redundancy number were determined, thus leading to reduced total system cost. The results confirmed that the combination method performed better than the three combination methods.

Gilvanejad *et al.* (2013a) presented a stochastic model for the electrical loads whereby the magnitude of the load demand was modelled using the normal cumulative distribution function. The results indicated that the only protective devices operated under the overload condition were those that protected the lower number of loads. For the case of a single protective device that protected several loads, the random load variation positive or negative, prevented the protective device from operating.

Gilvanejad *et al.* (2013b) noted that little attention had been given to the estimation of the occurrence rate of the cold-load pick up in distribution systems, which is a stochastic process. They developed a model that estimated the cold-load pick up peak magnitude of the outage time.

Rani *et al.* (2014) stated that system availability was an important issue. The objective of their study was to find the optimal policy for the mean time between failures (MTBF), the mean time to repair (MTTR) and related costs. They presented an application of Artificial Bees' Colony to solve the availability allocation of a series-parallel system, with a view to maximization of availability. In addition they obtained the minimum cost configuration of the system that met the given availability constraints.

Zoulfagharia *et al.* (2015) noted that reliability was one of the most important dependability attribute of electrical and mechanical systems, especially with applications in the space communication industries, internet networks, telecommunication systems, power generation systems, and productive facilities. They studied the availability optimization

of a system which had both repairable and non-repairable components. A new Mixed Integer Nonlinear Programming (MINLP) model was used in order to maximize the system availability and to minimize the total cost of the system. Empirical results indicated the satisfactory and appropriate system availability and the genetic algorithm was found to be precise.

Nguyen *et al.* (2016) used stochastic reward nets to model the availability of a data center for disaster tolerance. The model incorporated a two-level high availability configuration, fault and disaster tolerant techniques, dependencies between subsystems and disastrous events and failures experienced during data transmission. The results obtained showed that the disaster tolerant data center increased system availability .

3 THE MODELS

In sub-section 3.1, the CTMC availability model for a single exponentially distributed unplanned outage, Cao *et al.* (2002), is presented. In subsequent sections the non-birth-death CTMC availability models with exponentially and non-exponentially distributed unplanned outages are presented, Musiga *et al.* (in press).

3.1 Continuous Time Markov Chain Availability Model for A Single Exponentially Distributed Unplanned Outages

Cao *et al.* (2002) considered a system that was either in state 0 in which the system was working perfectly, in state 1 where the system was under repair or in state 2 in which the system was undergoing a scheduled upgrade, as shown in Figure 3.1. The failure and repair rates were λ and μ , respectively while the time to planned outage and time to upgrade were generally distributed with cumulative distribution functions $F(t)$ and $G(t)$.

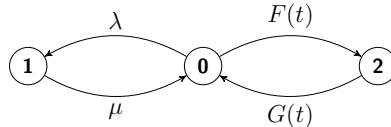


Figure 3.1. State diagram of a system with an unplanned and a planned outage.

For an exponentially distributed time to planned outage, the availability model of the

system was obtained as

$$A = \left(1 + \frac{\lambda}{\mu} + \frac{\lambda_m}{\mu_m} \right)^{-1} \quad (1)$$

In the subsequent sections λ_i and μ_i represent the failure and repair rates of type i .

3.2 Continuous Time Markov Chain Availability Model for Two Exponentially Distributed Unplanned Outages

Consider a system with two failed states 1 and 2, each of which follows an exponential distribution as shown in Figure 3.2.

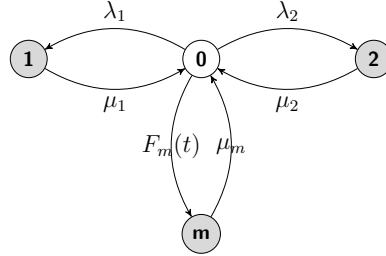


Figure 3.2. State diagram of a system with 2 exponentially distributed unplanned outages and a generally distributed planned outage.

The availability model of the system was obtained as

$$A = \left(1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_m}{\mu_m} \right)^{-1} \quad (2)$$

3.3 Continuous Time Markov Chain Availability Model for Three Exponentially Distributed and Three 2-Stage Hypoexponentially Distributed Unplanned Outages

Consider a system with six failed states 1, 2, 3, 4, 5 and 6, where failures into states 1, 2 and 3 are exponentially distributed and failures into states 4, 5 and 6 are 2-stage hypoexponentially distributed, as shown in Figure 3.3.

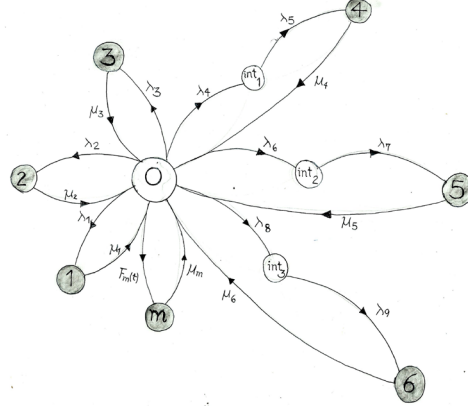


Figure 3.3 State diagram of a system with 3 exponentially and three 2-stage hypoexponentially distributed unplanned outages and a generally distributed planned outage.

The availability model of the system was obtained as

$$\begin{aligned}
 A = & \left\{ 1 + \frac{1}{(\lambda_5 - \lambda_4)(\lambda_7 - \lambda_6)(\lambda_9 - \lambda_8)} \left\{ \left(\prod_{i=1,2,3} \frac{\lambda_i}{\mu_i} \right) \left\{ \frac{\prod_{i=5,7,9} \lambda_i}{\sum_{i=1,2,3,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,6,8} \lambda_i} \right) \right. \right. \right. \\
 & - \frac{\prod_{i=5,7,8} \lambda_i}{\sum_{i=1,2,3,4,6,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,6,9} \lambda_i} \right) - \frac{\prod_{i=5,6,9} \lambda_i}{\sum_{i=1,2,3,4,7,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,7,8} \lambda_i} \right) \\
 & + \frac{\prod_{i=5,6,8} \lambda_i}{\sum_{i=1,2,3,4,7,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,7,9} \lambda_i} \right) - \frac{\prod_{i=4,7,9} \lambda_i}{\sum_{i=1,2,3,5,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,6,8} \lambda_i} \right) \\
 & + \frac{\prod_{i=4,7,8} \lambda_i}{\sum_{i=1,2,3,5,6,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,6,9} \lambda_i} \right) + \frac{\prod_{i=4,6,9} \lambda_i}{\sum_{i=1,2,3,5,7,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,7,8} \lambda_i} \right) \\
 & \left. \left. - \frac{\prod_{i=4,6,8} \lambda_i}{\sum_{i=1,2,3,5,7,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,7,9} \lambda_i} \right) \right\} \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{\prod_{i=4,5,7,9} \lambda_i}{\sum_{i=1,2,3,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,6,8} \lambda_i} \right) - \frac{\prod_{i=4,5,7,9} \lambda_i}{\sum_{i=1,2,3,5,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,6,8} \lambda_i} \right) \right. \\
& - \frac{\prod_{i=4,5,7,8} \lambda_i}{\sum_{i=1,2,3,4,6,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,6,9} \lambda_i} \right) + \frac{\prod_{i=4,5,7,8} \lambda_i}{\sum_{i=1,2,3,5,6,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,6,9} \lambda_i} \right) \\
& - \frac{\prod_{i=4,5,6,9} \lambda_i}{\sum_{i=1,2,3,4,7,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,7,8} \lambda_i} \right) + \frac{\prod_{i=4,5,6,9} \lambda_i}{\sum_{i=1,2,3,5,7,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,7,8} \lambda_i} \right) \\
& \left. + \frac{\prod_{i=4,5,6,8} \lambda_i}{\sum_{i=1,2,3,4,7,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,7,9} \lambda_i} \right) - \frac{\prod_{i=4,5,6,8} \lambda_i}{\sum_{i=1,2,3,5,7,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,7,9} \lambda_i} \right) \right\} \times \left\{ \frac{1}{\mu_4} \right\} \\
& + \left\{ \frac{\prod_{i=5,6,7,9} \lambda_i}{\sum_{i=1,2,3,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,6,8} \lambda_i} \right) - \frac{\prod_{i=5,6,7,9} \lambda_i}{\sum_{i=1,2,3,4,7,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,7,8} \lambda_i} \right) \right. \\
& - \frac{\prod_{i=5,6,7,8} \lambda_i}{\sum_{i=1,2,3,4,6,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,6,9} \lambda_i} \right) + \frac{\prod_{i=5,6,7,8} \lambda_i}{\sum_{i=1,2,3,4,7,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,7,9} \lambda_i} \right) \\
& - \frac{\prod_{i=4,6,7,9} \lambda_i}{\sum_{i=1,2,3,5,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,6,8} \lambda_i} \right) + \frac{\prod_{i=4,6,7,9} \lambda_i}{\sum_{i=1,2,3,5,7,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,7,8} \lambda_i} \right) \\
& \left. + \frac{\prod_{i=4,6,7,8} \lambda_i}{\sum_{i=1,2,3,5,6,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,6,9} \lambda_i} \right) - \frac{\prod_{i=4,6,7,8} \lambda_i}{\sum_{i=1,2,3,5,7,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,7,9} \lambda_i} \right) \right\} \times \left\{ \frac{1}{\mu_5} \right\} \\
& + \left\{ \frac{\prod_{i=5,7,8,9} \lambda_i}{\sum_{i=1,2,3,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,6,8} \lambda_i} \right) - \frac{\prod_{i=5,7,8,9} \lambda_i}{\sum_{i=1,2,3,4,6,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,6,9} \lambda_i} \right) \right. \\
& - \frac{\prod_{i=5,6,8,9} \lambda_i}{\sum_{i=1,2,3,4,7,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,7,8} \lambda_i} \right) + \frac{\prod_{i=5,6,8,9} \lambda_i}{\sum_{i=1,2,3,4,7,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,7,9} \lambda_i} \right) \\
& \left. - \frac{\prod_{i=4,7,8,9} \lambda_i}{\sum_{i=1,2,3,5,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,6,8} \lambda_i} \right) + \frac{\prod_{i=4,7,8,9} \lambda_i}{\sum_{i=1,2,3,5,6,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,6,9} \lambda_i} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\prod_{i=4,6,8,9} \lambda_i}{\sum_{i=1,2,3,5,7,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,7,8} \lambda_i} \right) - \frac{\prod_{i=4,6,8,9} \lambda_i}{\sum_{i=1,2,3,5,7,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,7,9} \lambda_i} \right) \Bigg\} \times \left\{ \frac{1}{\mu_6} \right\} \\
& + \left\{ \left(\frac{\lambda_m \prod_{i=5,7,9} \lambda_i}{\lambda_m + \sum_{i=1,2,3,4,6,8} \lambda_i} \right) - \left(\frac{\lambda_m \prod_{i=5,7,8} \lambda_i}{\lambda_m + \sum_{i=1,2,3,4,6,9} \lambda_i} \right) - \left(\frac{\lambda_m \prod_{i=5,6,9} \lambda_i}{\lambda_m + \sum_{i=1,2,3,4,7,8} \lambda_i} \right) \right. \\
& + \left(\frac{\lambda_m \prod_{i=5,6,8} \lambda_i}{\lambda_m + \sum_{i=1,2,3,4,7,9} \lambda_i} \right) - \left(\frac{\lambda_m \prod_{i=4,7,9} \lambda_i}{\lambda_m + \sum_{i=1,2,3,5,6,8} \lambda_i} \right) + \left(\frac{\lambda_m \prod_{i=4,7,8} \lambda_i}{\lambda_m + \sum_{i=1,2,3,5,6,9} \lambda_i} \right) \\
& + \left. \left(\frac{\lambda_m \prod_{i=4,6,9} \lambda_i}{\lambda_m + \sum_{i=1,2,3,5,7,8} \lambda_i} \right) - \left(\frac{\lambda_m \prod_{i=4,6,8} \lambda_i}{\lambda_m + \sum_{i=1,2,3,5,7,9} \lambda_i} \right) \right\} \left\{ \frac{1}{\mu_m} \right\} \\
& \div \left\{ \frac{\prod_{i=5,7,9} \lambda_i}{\sum_{i=1,2,3,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,6,8} \lambda_i} \right) - \frac{\prod_{i=5,7,8} \lambda_i}{\sum_{i=1,2,3,4,6,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,6,9} \lambda_i} \right) \right. \\
& - \frac{\prod_{i=5,6,9} \lambda_i}{\sum_{i=1,2,3,4,7,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,7,8} \lambda_i} \right) + \frac{\prod_{i=5,6,8} \lambda_i}{\sum_{i=1,2,3,4,7,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,4,7,9} \lambda_i} \right) \\
& - \frac{\prod_{i=4,7,9} \lambda_i}{\sum_{i=1,2,3,5,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,6,8} \lambda_i} \right) + \frac{\prod_{i=4,7,8} \lambda_i}{\sum_{i=1,2,3,5,6,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,6,9} \lambda_i} \right) \\
& + \left. \frac{\prod_{i=4,6,9} \lambda_i}{\sum_{i=1,2,3,5,7,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,7,8} \lambda_i} \right) - \frac{\prod_{i=4,6,8} \lambda_i}{\sum_{i=1,2,3,5,7,9} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,2,3,5,7,9} \lambda_i} \right) \right\}^{-1}
\end{aligned} \tag{3}$$

3.4 Continuous Time Markov Chain Availability Model for Four 2-Stage Hypoexponentially Distributed Unplanned Outages

Consider a system with four failed states 1, 2, 3 and 4, in two stages as shown in Figure 3.4.

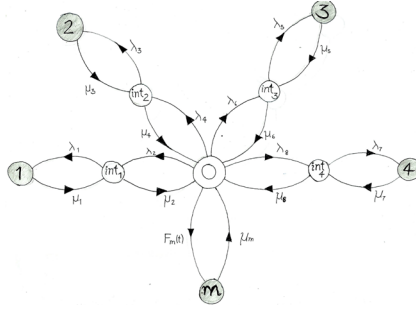


Figure 3.4 State diagram of a system with four 2-stage hypoexponentially distributed unplanned outages and a generally distributed planned outage.

The availability model of the system was obtained as

$$\begin{aligned}
 A = & \left\{ 1 + \frac{1}{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)(\lambda_6 - \lambda_5)(\lambda_8 - \lambda_7)} \left\{ \frac{\prod_{i=1,2,4,6,8} \lambda_i}{\sum_{i=1,3,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,5,7} \lambda_i} \right) \right. \right. \\
 & - \frac{\prod_{i=1,2,4,6,8} \lambda_i}{\sum_{i=2,3,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,5,7} \lambda_i} \right) - \frac{\prod_{i=1,2,4,6,7} \lambda_i}{\sum_{i=1,3,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,5,8} \lambda_i} \right) \\
 & + \frac{\prod_{i=1,2,4,6,7} \lambda_i}{\sum_{i=2,3,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,5,8} \lambda_i} \right) - \frac{\prod_{i=1,2,4,5,8} \lambda_i}{\sum_{i=1,3,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,6,7} \lambda_i} \right) \\
 & + \frac{\prod_{i=1,2,4,5,8} \lambda_i}{\sum_{i=2,3,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,6,7} \lambda_i} \right) + \frac{\prod_{i=1,2,4,5,7} \lambda_i}{\sum_{i=1,3,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,6,8} \lambda_i} \right) \\
 & - \frac{\prod_{i=1,2,4,5,7} \lambda_i}{\sum_{i=2,3,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,6,8} \lambda_i} \right) - \frac{\prod_{i=1,2,3,6,8} \lambda_i}{\sum_{i=1,4,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,5,7} \lambda_i} \right) \\
 & \left. - \frac{\prod_{i=1,2,3,6,8} \lambda_i}{\sum_{i=2,4,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,5,7} \lambda_i} \right) - \frac{\prod_{i=1,2,3,6,7} \lambda_i}{\sum_{i=1,4,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,5,8} \lambda_i} \right) \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\prod_{i=1,2,3,6,7} \lambda_i}{\sum_{i=2,4,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,5,8} \lambda_i} \right) - \frac{\prod_{i=1,2,3,5,8} \lambda_i}{\sum_{i=1,4,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,6,7} \lambda_i} \right) \\
& + \frac{\prod_{i=1,2,3,5,8} \lambda_i}{\sum_{i=2,4,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,6,7} \lambda_i} \right) + \frac{\prod_{i=1,2,3,5,7} \lambda_i}{\sum_{i=1,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,6,8} \lambda_i} \right) \\
& - \frac{\prod_{i=1,2,3,5,7} \lambda_i}{\sum_{i=2,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,6,8} \lambda_i} \right) \left. \right\} \times \left\{ \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \right\} \\
& + \frac{\prod_{i=2,3,4,6,8} \lambda_i}{\sum_{i=1,3,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,5,7} \lambda_i} \right) - \frac{\prod_{i=1,2,3,4,6,8} \lambda_i}{\sum_{i=1,4,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,5,7} \lambda_i} \right) \\
& - \frac{\prod_{i=2,3,4,6,7} \lambda_i}{\sum_{i=1,3,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,5,8} \lambda_i} \right) + \frac{\prod_{i=2,3,4,6,7} \lambda_i}{\sum_{i=1,4,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,5,8} \lambda_i} \right) \\
& - \frac{\prod_{i=2,3,4,5,8} \lambda_i}{\sum_{i=1,3,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,6,7} \lambda_i} \right) + \frac{\prod_{i=2,3,4,5,8} \lambda_i}{\sum_{i=1,4,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,6,7} \lambda_i} \right) \\
& + \frac{\prod_{i=2,3,4,5,7} \lambda_i}{\sum_{i=1,3,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,6,8} \lambda_i} \right) - \frac{\prod_{i=2,3,4,5,7} \lambda_i}{\sum_{i=1,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,6,8} \lambda_i} \right) \\
& - \frac{\prod_{i=1,3,4,6,8} \lambda_i}{\sum_{i=2,3,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,5,7} \lambda_i} \right) + \frac{\prod_{i=1,3,4,6,8} \lambda_i}{\sum_{i=2,4,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,5,7} \lambda_i} \right) \\
& + \frac{\prod_{i=1,3,4,6,7} \lambda_i}{\sum_{i=2,3,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,5,8} \lambda_i} \right) - \frac{\prod_{i=1,3,4,6,7} \lambda_i}{\sum_{i=2,4,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,5,8} \lambda_i} \right) \\
& + \frac{\prod_{i=1,3,4,5,8} \lambda_i}{\sum_{i=2,3,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,6,7} \lambda_i} \right) - \frac{\prod_{i=1,3,4,5,8} \lambda_i}{\sum_{i=2,4,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,6,7} \lambda_i} \right) \\
& - \frac{\prod_{i=1,3,4,5,7} \lambda_i}{\sum_{i=2,3,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,6,8} \lambda_i} \right) + \frac{\prod_{i=1,3,4,5,7} \lambda_i}{\sum_{i=2,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,6,8} \lambda_i} \right) \left. \right\} \times \left\{ \frac{\mu_3 + \mu_4}{\mu_3 \mu_4} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\prod_{i=2,4,5,6,8} \lambda_i}{\sum_{i=1,3,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,5,7} \lambda_i} \right) - \frac{\prod_{i=2,4,5,6,8} \lambda_i}{\sum_{i=1,3,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,6,7} \lambda_i} \right) \\
& - \frac{\prod_{i=2,4,5,6,7} \lambda_i}{\sum_{i=1,3,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,5,8} \lambda_i} \right) + \frac{\prod_{i=2,4,5,6,7} \lambda_i}{\sum_{i=1,3,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,6,8} \lambda_i} \right) \\
& - \frac{\prod_{i=2,3,5,6,8} \lambda_i}{\sum_{i=1,4,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,5,7} \lambda_i} \right) + \frac{\prod_{i=2,3,5,6,8} \lambda_i}{\sum_{i=1,4,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,6,7} \lambda_i} \right) \\
& + \frac{\prod_{i=2,3,5,6,7} \lambda_i}{\sum_{i=1,4,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,5,8} \lambda_i} \right) - \frac{\prod_{i=2,3,5,6,7} \lambda_i}{\sum_{i=1,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,6,8} \lambda_i} \right) \\
& - \frac{\prod_{i=1,4,5,6,8} \lambda_i}{\sum_{i=2,3,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,5,7} \lambda_i} \right) + \frac{\prod_{i=1,4,5,6,8} \lambda_i}{\sum_{i=2,3,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,6,7} \lambda_i} \right) \\
& + \frac{\prod_{i=1,4,5,6,7} \lambda_i}{\sum_{i=2,3,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,5,8} \lambda_i} \right) - \frac{\prod_{i=1,4,5,6,7} \lambda_i}{\sum_{i=2,3,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,6,8} \lambda_i} \right) \\
& + \frac{\prod_{i=1,3,5,6,7} \lambda_i}{\sum_{i=2,4,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,5,7} \lambda_i} \right) - \frac{\prod_{i=1,3,5,6,7} \lambda_i}{\sum_{i=2,4,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,6,7} \lambda_i} \right) \\
& - \frac{\prod_{i=1,3,5,6,7} \lambda_i}{\sum_{i=2,4,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,5,8} \lambda_i} \right) + \frac{\prod_{i=1,3,5,6,7} \lambda_i}{\sum_{i=2,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,6,8} \lambda_i} \right) \Bigg\} \times \left\{ \frac{\mu_5 + \mu_6}{\mu_5 \mu_6} \right\} \\
& + \frac{\prod_{i=2,4,6,7,8} \lambda_i}{\sum_{i=1,3,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,5,7} \lambda_i} \right) - \frac{\prod_{i=2,4,6,7,8} \lambda_i}{\sum_{i=1,3,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,5,8} \lambda_i} \right) \\
& - \frac{\prod_{i=2,4,5,7,8} \lambda_i}{\sum_{i=1,3,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,6,7} \lambda_i} \right) + \frac{\prod_{i=2,4,5,7,8} \lambda_i}{\sum_{i=1,3,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,6,8} \lambda_i} \right) \\
& - \frac{\prod_{i=2,3,6,7,8} \lambda_i}{\sum_{i=1,4,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,5,7} \lambda_i} \right) + \frac{\prod_{i=2,3,6,7,8} \lambda_i}{\sum_{i=1,4,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,5,8} \lambda_i} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\prod_{i=2,3,5,7,8} \lambda_i}{\sum_{i=1,4,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,6,7} \lambda_i} \right) - \frac{\prod_{i=2,3,5,7,8} \lambda_i}{\sum_{i=1,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,6,8} \lambda_i} \right) \\
& - \frac{\prod_{i=1,4,6,7,8} \lambda_i}{\sum_{i=2,3,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,5,7} \lambda_i} \right) + \frac{\prod_{i=1,4,6,7,8} \lambda_i}{\sum_{i=2,3,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,5,8} \lambda_i} \right) \\
& + \frac{\prod_{i=1,4,5,7,8} \lambda_i}{\sum_{i=2,3,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,6,7} \lambda_i} \right) - \frac{\prod_{i=1,4,5,7,8} \lambda_i}{\sum_{i=2,3,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,6,8} \lambda_i} \right) \\
& + \frac{\prod_{i=1,3,6,7,8} \lambda_i}{\sum_{i=2,4,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,5,7} \lambda_i} \right) - \frac{\prod_{i=1,3,6,7,8} \lambda_i}{\sum_{i=2,4,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,5,8} \lambda_i} \right) \\
& - \frac{\prod_{i=1,3,5,7,8} \lambda_i}{\sum_{i=2,4,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,6,7} \lambda_i} \right) + \frac{\prod_{i=1,3,5,7,8} \lambda_i}{\sum_{i=2,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,6,8} \lambda_i} \right) \Big\} \times \left\{ \frac{\mu_7 + \mu_8}{\mu_7 \mu_8} \right\} \\
& + \left\{ \left(\frac{\lambda_m \prod_{i=2,4,6,8} \lambda_i}{\lambda_m + \sum_{i=1,3,5,7} \lambda_i} \right) - \left(\frac{\lambda_m \prod_{i=2,4,6,7} \lambda_i}{\lambda_m + \sum_{i=1,3,5,8} \lambda_i} \right) - \left(\frac{\lambda_m \prod_{i=2,4,5,8} \lambda_i}{\lambda_m + \sum_{i=1,3,6,7} \lambda_i} \right) + \left(\frac{\lambda_m \prod_{i=2,4,5,7} \lambda_i}{\lambda_m + \sum_{i=1,3,6,8} \lambda_i} \right) \right. \\
& - \left(\frac{\lambda_m \prod_{i=2,3,6,8} \lambda_i}{\lambda_m + \sum_{i=1,4,5,7} \lambda_i} \right) + \left(\frac{\lambda_m \prod_{i=2,3,6,7} \lambda_i}{\lambda_m + \sum_{i=1,4,5,8} \lambda_i} \right) + \left(\frac{\lambda_m \prod_{i=2,3,5,8} \lambda_i}{\lambda_m + \sum_{i=1,4,6,7} \lambda_i} \right) - \left(\frac{\lambda_m \prod_{i=2,3,5,7} \lambda_i}{\lambda_m + \sum_{i=1,4,6,8} \lambda_i} \right) \\
& - \left(\frac{\lambda_m \prod_{i=1,4,6,8} \lambda_i}{\lambda_m + \sum_{i=2,3,5,7} \lambda_i} \right) + \left(\frac{\lambda_m \prod_{i=1,4,6,7} \lambda_i}{\lambda_m + \sum_{i=2,3,5,8} \lambda_i} \right) + \left(\frac{\lambda_m \prod_{i=1,4,5,8} \lambda_i}{\lambda_m + \sum_{i=2,3,6,7} \lambda_i} \right) - \left(\frac{\lambda_m \prod_{i=1,4,5,7} \lambda_i}{\lambda_m + \sum_{i=2,3,6,8} \lambda_i} \right) \\
& \left. + \left(\frac{\lambda_m \prod_{i=1,3,6,8} \lambda_i}{\lambda_m + \sum_{i=2,4,5,7} \lambda_i} \right) - \left(\frac{\lambda_m \prod_{i=1,3,6,7} \lambda_i}{\lambda_m + \sum_{i=2,4,5,8} \lambda_i} \right) - \left(\frac{\lambda_m \prod_{i=1,3,5,8} \lambda_i}{\lambda_m + \sum_{i=2,4,6,7} \lambda_i} \right) + \left(\frac{\lambda_m \prod_{i=1,3,5,7} \lambda_i}{\lambda_m + \sum_{i=2,4,6,8} \lambda_i} \right) \right\} \left\{ \frac{1}{\mu_m} \right\} \\
& \div \left\{ \frac{\prod_{i=2,4,6,8} \lambda_i}{\sum_{i=1,3,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,5,7} \lambda_i} \right) - \frac{\prod_{i=2,4,6,7} \lambda_i}{\sum_{i=1,3,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,5,8} \lambda_i} \right) \right. \\
& \left. - \frac{\prod_{i=2,4,5,8} \lambda_i}{\sum_{i=1,3,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,6,7} \lambda_i} \right) + \frac{\prod_{i=2,4,5,7} \lambda_i}{\sum_{i=1,3,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,3,6,8} \lambda_i} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\prod_{i=2,3,6,8} \lambda_i}{\sum_{i=1,4,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,5,7} \lambda_i} \right) + \frac{\prod_{i=2,3,6,7} \lambda_i}{\sum_{i=1,4,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,5,8} \lambda_i} \right) \\
& + \frac{\prod_{i=2,3,5,8} \lambda_i}{\sum_{i=1,4,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,6,7} \lambda_i} \right) - \frac{\prod_{i=2,3,5,7} \lambda_i}{\sum_{i=1,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=1,4,6,8} \lambda_i} \right) \\
& - \frac{\prod_{i=1,4,6,8} \lambda_i}{\sum_{i=2,3,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,5,7} \lambda_i} \right) + \frac{\prod_{i=1,4,6,7} \lambda_i}{\sum_{i=2,3,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,5,8} \lambda_i} \right) \\
& + \frac{\prod_{i=1,4,5,8} \lambda_i}{\sum_{i=2,3,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,6,7} \lambda_i} \right) - \frac{\prod_{i=1,4,5,7} \lambda_i}{\sum_{i=2,3,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,3,6,8} \lambda_i} \right) \\
& + \frac{\prod_{i=1,3,6,8} \lambda_i}{\sum_{i=2,4,5,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,5,7} \lambda_i} \right) - \frac{\prod_{i=1,3,6,7} \lambda_i}{\sum_{i=2,4,5,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,5,8} \lambda_i} \right) \\
& - \frac{\prod_{i=1,3,5,8} \lambda_i}{\sum_{i=2,4,6,7} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,6,7} \lambda_i} \right) + \frac{\prod_{i=1,3,5,7} \lambda_i}{\sum_{i=2,4,6,8} \lambda_i} \left(1 - \frac{\lambda_m}{\lambda_m + \sum_{i=2,4,6,8} \lambda_i} \right) \Bigg\}^{-1} \quad (4)
\end{aligned}$$

4 SYSTEM DESCRIPTION OF CISCO 12000 GSR

Cisco 12000 GSR is an electronic device that connects networks and directs data transfer from source to destination, Trivedi *et al.* (2012). The modules of Cisco 12000 GSR and their functions include: (i) the Gigabit Route Processor (GRP) which determines the network topology and calculates the best path across the network (ii) Line Cards (LC) that forward data packets (iii) Clock and Scheduler Cards (CSC) which accept transmission requests from Line Cards and provide reference clock for synchronization and (iv) Switch Fabric Cards (SFC) that receive scheduling information and clocking reference from the Clock and Scheduler Cards, then perform switching functions.

The high availability features in Cisco 12000 GSR include: (i) Gigabit Switch Router redundancy where a 1 active and 1 standby redundant scheme is used (ii) Clock and Scheduler Card/Switch Fabric Card redundancy where, though the Cisco 12000 GSR requires at least one Clock and Scheduler Card, an additional Clock and Scheduler Card

is availed (iii) Switch Fabric Cards where for every N active Switch Fabric Cards there is a standby Switch Fabric Card (iv) Power supply and cooling redundancy, Cisco 12000 GSR requires at least two power supplies and (v) Online insertion and removal such that the replacement of faulty modules is supported for all cards without affecting system operation. Two assumptions were made, namely that any upgrade caused system outage and the power supply was a component of the chassis. The configuration of Cisco 12000 GSR is shown in Table 4.1.

Module	Redundancy	Types of Outages
LC-in	1	Hardware,Software
LC-out	1	Hardware,Software
CSC	2	Hardware
SFC	3	Hardware
GRP	1:1	Hardware
Chassis	1	Hardware
IOS	1:1	Software

Table 4.1 Configuration of Cisco 12000 GSR.

The Chassis, Line Card-in and Line Card-out modules are non-redundant modules while the Gigabit Route Processor and the IOS Software modules are 1:1, one primary and one standby redundant modules. There are two Clock and Scheduler Cards and three Switch Fabric Cards. Cisco 12000 GSR is available when at least four out of five Clock and Scheduler Card/Switch Fabric Card modules are functional. The Clock and Scheduler Card, Switch Fabric Card, IOS Software, Gigabit Route Processor and Chassis have one type of outage, either hardware or software failure only while the Line Card-in and Line Card-out have two types of outages, both hardware and software failures. Cisco 12000 GSR is a hierarchical engineering system as depicted in Figure 4.1. It comprises of modules which are connected in series while the system undergoes diverse states within the modules. Thus at the higher level is a reliability block diagram where each block corresponds to a module of the system as shown in Figure 4.1.

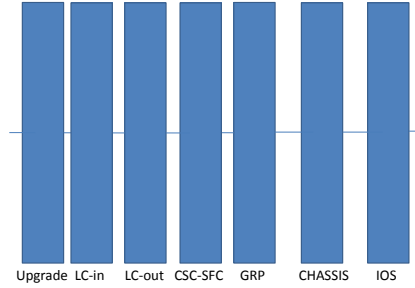


Figure 4.1 Reliability Block Diagram for Cisco 12000 GSR.

5 CTMC MODELS FOR MODULES OF CISCO 12000 GSR

According to Trivedi *et al.* (2012), the lower level of Cisco 12000 GSR was modelled using CTMCs which captured the up and down states within the individual modules, as replicated in the following subsections.

5.1 CTMC Model for Upgrade and Chassis Modules

The upgrade and chassis modules have two states, state up where the system is in good functioning condition and state down where the system has failed, hence were modeled as a two-state CTMC as shown in Figure 5.1.

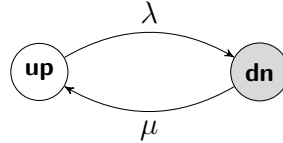


Figure 5.1 CTMC model for non-redundant module with one kind of outage and recovery.

5.2 CTMC Model for Line Card-In and Line Card-Out Modules

The Line Card-in and Line Card-out modules have two kinds of outages and recoveries, software and hardware failures and repairs, hence were modeled as a three-state CTMC as shown in Figure 5.2.

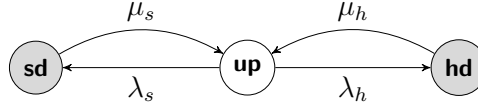


Figure 5.2 CTMC model for non-redundant module with two kinds of outages and recoveries.

The CTMC model in Figure 5.2 has three states, state up where the system is in perfect functioning condition, state software-down (sd) where the software has failed hence the system is down and state hardware-down (hd) where the hardware has failed hence the system is down.

5.3 CTMC Model for Gigabit Route Processor and IOS Software Modules

The Gigabit Route Processor and IOS Software modules are 1:N redundant modules with N active and 1 standby units. The up states and down states as well as imperfect coverage and switch-over factor are considered. The CTMC model is shown in Figure 5.3. The model has states 1, 9, 10, 11, 12, 13, 14 and 15 out of which states 12, 13 and 15 are down states. In state 1 all the components are functioning. Failures are either detected or not detected, and with detection switch-over occurs to standby units.

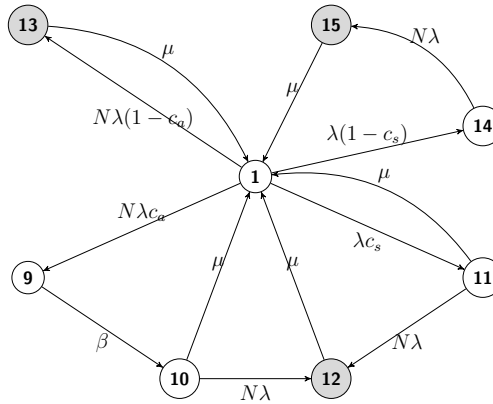


Figure 5.3 CTMC model for 1:N redundant module.

5.4 CTMC Model for Clock and Scheduler Card/Switch Fabric Card Module

The Clock and Scheduler Card/Switch Fabric Card Module is available when at least four out of five Clock and Scheduler Card/Switch Fabric Cards are functional. The CTMC model of the Clock and Scheduler Card/Switch Fabric Card module is shown in Figure 5.4. The states in Figure 5.4 which are represented by the pair $m - n$ imply that m Clock and Scheduler Cards and n Switch Fabric Cards are functional. The up states are $2 - 3, 1 - 3$ and $2 - 2$ since at least four cards are functioning. The down states are $0 - 3, 1 - 2$ and $2 - 1$, here less than four cards are functioning.

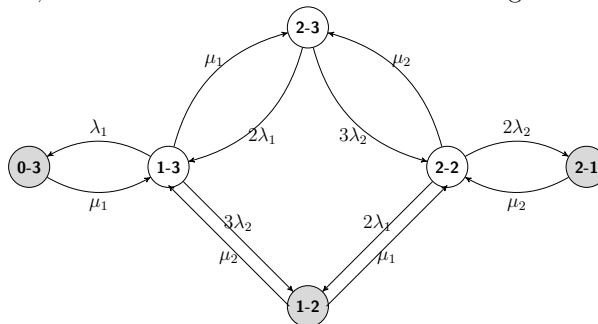


Figure 5.4 CTMC model for Clock Scheduler Card/Switch Fabric Card module.

6 APPLICATION

In this section, the non-birth-death CTMC availability models with exponentially and non-exponentially distributed unplanned outages is applied to Cisco 12000 GSR. The input parameter values are obtained from Lee F. (2000). The types of failure and repair distributions experienced in each of the modules are considered and the state of planned outage is introduced into each of the module models. The upgrade module is considered as a planned outage since the process of upgrading a system is a planned outage conducted by the engineer.

6.1 Chassis Module

In the chassis module CTMC model shown in Figure 5.1, there is a single exponentially distributed unplanned outage and recovery thus introducing the planned outage state m , then the model is a three-state CTMC as shown in Figure 6.1.

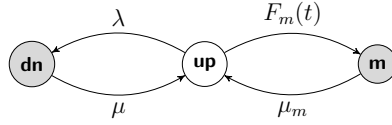


Figure 6.1 CTMC model for non-redundant module with an exponentially distributed unplanned outage, a recovery and a planned outage.

From the upgrade module, the repair rate $\mu_m = 2$ and the failure rate $\lambda_m = 4320^{-1}$, while for the chassis module the failure rate is 396510^{-1} and the repair rate is 2. Substituting these values in equation (1), the availability of the chassis module is obtained as 0.999883011.

6.2 Line Card-In and Line Card-Out Modules

In the Line Card-in and Line Card-out modules CTMC model shown in Figure 5.2, there are two exponentially distributed unplanned outages and recoveries, that is software and hardware failures and repairs, hence introducing the planned outage state m then the model is a four-state CTMC as shown in Figure 6.2.

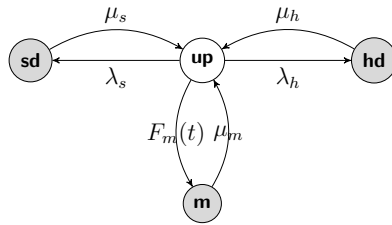


Figure 6.2 CTMC model for non-redundant module with two exponentially distributed unplanned outages, recoveries and a planned outage.

For the Line Card-In module, the software failure rate is 18000^{-1} , the software repair rate is 2, the hardware failure rate is 111050^{-1} and the hardware repair rate is 0.5. Substituting these values in equation (2), the availability of the Line Card-In module is obtained as 0.999838497.

Similarly for the Line Card-Out module where the software failure rate is 18000^{-1} , the software repair rate is 2, the hardware failure rate 103402^{-1} and the hardware repair rate is 0.5. On substituting the values into equation (2), the availability of the Line Card-Out module is obtained as 0.999837166.

6.3 Gigabit Route Processor and IOS Software Modules

The Gigabit Route Processor and IOS Software modules CTMC model shown in Figure 5.3 shows that the modules undergo three exponentially distributed and three 2-stage hypoexponentially distributed unplanned outages with recoveries. Introducing the planned outage state m then the model is a nine-state CTMC as shown in Figure 6.3.

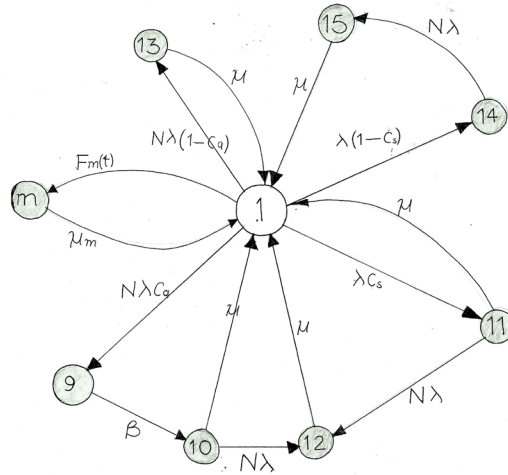


Figure 6.3 CTMC model for 1:N redundant module with three exponentially and three 2-stage hypoexponentially distributed unplanned outages, recoveries and a planned outage.

State m which is a planned outage is applicable to the Gigabit Route Processor module only since an upgrade of IOS software is the installation of a new software. In the Gigabit

Route Processor module, $N = 1$, the failure rate is 108304^{-1} , the repair rate is $= 1$, the coverage of an active unit is 0.9 and the coverage of a standby unit is 0.6. Let $\lambda_1 = 9.2333 \times 10^{-7}$, $\lambda_2 = 5.5400 \times 10^{-6}$, $\lambda_3 = 8.3099 \times 10^{-6}$, $\lambda_4 = 5.5400 \times 10^{-6}$, $\lambda_5 = 9.2333 \times 10^{-6}$, $\lambda_6 = 3.6933 \times 10^{-6}$, $\lambda_7 = 9.2333 \times 10^{-6}$, $\lambda_8 = 8.3099 \times 10^{-6}$, $\lambda_9 = 9.2333 \times 10^{-6}$ and $\mu_i = 1, i = 1, \dots, 6$. Substituting these values in equation (3), the availability of the Gigabit Route Processor module is obtained as 0.999868001.

6.4 Clock and Scheduler Card/ Switch Fabric Card Module

The Clock and Scheduler Card/Switch Fabric Card module undergoes four 2-stage hypoexponentially distributed unplanned outages with recoveries as shown in Figure 5.4. Introducing the planned outage state m , then the model is a seven-state CTMC as shown in Figure 6.4.

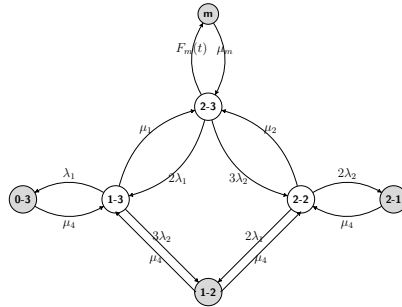


Figure 6.4 CTMC model for Clock Scheduler Card/Switch Fabric Card Module with four 2-stage hypoexponentially distributed unplanned outages, recoveries and a planned outage.

The failure and repair rates in the Clock and Scheduler Card are 272584^{-1} and 0.5 while the failure and repair rates in the Switch Fabric Card are 422115^{-1} and 0.5, respectively. Let $\lambda_1 = 3.6686 \times 10^{-6}$, $\lambda_2 = 7.3372 \times 10^{-6}$, $\lambda_3 = 7.1071 \times 10^{-6}$, $\lambda_4 = 7.3372 \times 10^{-6}$, $\lambda_5 = 7.1071 \times 10^{-6}$, $\lambda_6 = 7.3372 \times 10^{-6}$, $\lambda_7 = 4.7380 \times 10^{-6}$, $\lambda_8 = 7.1071 \times 10^{-6}$ and $\mu_i = 0.5, i = 1, \dots, 8$. Substituting these values in equation (4), the availability of the Clock Scheduler Card/Switch Fabric Card module is obtained as 0.99991559.

6.5 Availability of Cisco 12000 GSR with Exponentially and Hypoexponentially distributed unplanned outages

Cisco 12000 Gigabit Switch Router is connected in series thus the availability model is given by the product expression

$$A(Cisco) = A(cha) \times A(lc - in) \times A(lc - out) \times A(grp) \times A(csc/sfc) \quad (5)$$

Thus substituting the computed availabilities of the individual modules into equation (5), the availability of Cisco 12000 Gigabit Switch Router, with the incorporation of exponentially and hypoexponentially distributed unplanned outages, is obtained as 0.999342435.

7 CONCLUSIONS

In this work, the non-birth-death CTMC availability models with exponentially and non-exponentially distributed unplanned outages was applied to Cisco 12000 GSR. It was found that the availability of the system is 0.999342435, which compares favourably to the steady-state availability of 0.99978681.

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Efficiencies of D-,A-,E- and T-Optimal Designs for a Second Order Rotatable Design in four Dimensions Constructed using Balanced Incomplete Block Designs.

1. Department of Physical and Mathematical Sciences, School of pure and applied Sciences, Mt. Kenya University, Thika, Kenya
2. Department of Mathematics, Physics and Computing, School of Sciences and Aerospace Studies, Moi University Eldoret, Kenya

Email address:

kabuegichuki@yahoo.com (K. T. Gichuki), koske4@yahoo.co.uk
(K. Joseph), johnkasome@yahoo.com (M. John)

Abstract

The best experimental design in any given condition is the one which estimates the desired effects and contrasts with maximum precision or efficiency. Efficiency as discriminating criteria allows the comparison between any design and the best design. One of the main objectives of response surface methodology (RSM) is the determination of the optimum settings of the control variables that result in a maximum (or a minimum) response over a certain region of interest, R. This requires having a 'good' fitting model that provides an adequate representation of the mean response because such a model is to be utilized to determine the value of the optimum .D-,A-,E-and T- Optimal designs of a second order rotatable design in four dimensions constructed using balanced incomplete block designs (BIBD) when the number of replications are less than three times number of pairs of treatments occur together and their relative efficiencies to General designs are presented in this paper E-optimal design was found to be the most efficient design with only 32 runs.

Keywords: *Relative Efficiency, Optimal Designs.*

1.0 INTRODUCTION:

Optimal designs are designs of fewer trials than non-optimal designs which are carried out with the aim of obtaining an efficient design for fitting a reduced quadratic or higher order model and they are the alternative designs when the experimental region is irregular due to factor levels constraints or when the experimenter has prior knowledge about the process being studied which may suggest a non-standard model where some higher order terms or some interaction terms between factors may not be included in the model and the interest is to obtain an efficient design [1]. Optimal designs have been suggested and are used frequently in practice [2]. The best design to use in any given situation is the one which provides the estimate of desired effects and contrasts with maximum precision (efficiency), and has a simple layout and analysis [3]. Selecting an appropriate experimental design, is based on finding the best optimality criteria in which larger efficiency values imply a better design, [4]. Response surface methodology based on D-optimal design was employed to study the significance and interactive effect of methanol-to-oil (M:O) molar ratio, catalyst concentration, reaction time, and mixing rate on biodiesel yield, [5]. The efficiency of an experiment is influenced by the adoption of an appropriate experimental design capable of representing the response surface design. Reference [6] studied the measure of efficiency of the design matrix under Latin squares and orthogonality properties of designs while [7] reviewed some fundamentals of experimental design in particular orthogonality and balance, and introduced the idea of design efficiency by comparing some widely available design software including Sawtooth Software's CVA and SAS Institute's OPTEX programs. Reference [8] demonstrated the use of Efficient design of response surface methodology as a method to optimize experiments so as to capture better synergies between species and conditions. Their work indicated the suitability of the approach to model carbon dioxide corrosion at pH 4–5.5

2.0 Methodology:

According to [6], the efficiency and sensitivity of a design may be very much affected by the choice of the design matrix X . Given $S_{p \times p} = X'X$ a real non-

negative symmetric matrix of rank $r \leq p$. Let t be a column vector with p components not all zero such that the equation $S_p = t$ has solutions for p . If p is any solution for $S_p = t$ then the inequality

$$\frac{1}{\lambda_{max}} \leq \frac{p'S_p}{t't} \leq \frac{1}{\lambda_{min}} \quad (1)$$

holds where λ_{max} and λ_{min} are the non-zero characteristics roots of S . If we denote with u , the maximum value of λ_{min} for x_{ai} in T such that T is an orthogonal matrix satisfying

$$T'AT = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_r \end{bmatrix} \quad (2)$$

Where A is a $k \times k$ matrix such that

$$S^{-1} = \begin{bmatrix} A & B \\ B' & D \end{bmatrix}. \quad (3)$$

Now when S is of full rank, equation (1) becomes

$$\frac{1}{\lambda_{max}} \leq \frac{t'S^{-1}t}{t't} \leq \frac{1}{\lambda_{min}} \quad (4)$$

For testing linear hypothesis of the regression coefficient $H_0: \beta_1 = \beta_2 = \cdots = \beta_r = 0$ when $r \leq p$ for a model, the efficiency is a ratio $\frac{\lambda_{min}}{u}$ and the design is most efficient if the efficiency of the design is equal to one [9]. Reference [10] gave details of determination of the moment matrix and the D-, A-, E- and T-optimal values of a general second order rotatable design in four dimensions constructed using balanced incomplete blocks design when the number of replications (r) are less than three the number of times (λ) pairs of treatments occur together in the design as put forward by [11] as 0.6796529, 0.04104631, 0.002856958 and 1.135448 respectively, when all the regression vectors are run an equal number of times, we have a uniform design. A design can be made better by varying the proportion that a particular vector is run. Reference [12] gave details of obtaining the optimal weights of a design for matrix means ϕ_p with $p \in (-\infty, 1]$ where the optimal weights satisfy $w_i = \frac{\sqrt{b_{ii}}}{\sum_{j \leq N} \sqrt{b_{jj}}} \forall i = 1, \dots, N$

$$(5)$$

Where $b_{11}, \dots, b_{NN}$ are the diagonal elements of matrix B given as equation (6)

$$B = UC^{p+1}U' \quad (6)$$

Where $U = (XX')^{-1}XK$ and C is the information matrix, K is the coefficient matrix and the N regression vectors $x_1, \dots, x_N$ form the rows of the design matrix X . Hence w_i will form the proportion that each regression vector will be run to

$$V(\emptyset_p) = \left(\frac{1}{S} \text{trace } C^p\right)^{1/p} = \left(\frac{1}{S} (\sum_{j \leq N} \sqrt{b_{jj}})^2\right)^{1/p} \quad \text{if } p \neq 0. \quad \text{For } p \in (-\infty, 1) \quad (7)$$

2.1 D-Optimal Efficiency

To obtain optimal weights of D-optimal design equation (5) is used after putting $P = 0$ to obtain matrix B in equation (6) where $p \in (-\infty, 1)$ and the corresponding optimal variance from equation (7). B_d Matrix is given as

$$B_d = UCU' \quad (8)$$

Factorial weight corresponding to D -optimal $D_{fw} = \frac{\sqrt{b_{11}}}{\sum_{j \leq N} \sqrt{b_{jj}}}$ while the weight corresponding to axial part is obtained as $D_{aw} = \frac{\sqrt{b_{NN}}}{\sum_{j \leq N} \sqrt{b_{jj}}}$ Where $b_{11}, \dots, b_{NN}$ are the diagonal elements of matrix B_d . The D -optimal design is obtained by replicating the factorial and axial parts as per the weights obtained to give the design matrix X_d which is used to obtain the D-optimal moment matrix $C_d = \frac{X_d' X_d}{N_d}$ where N_d is the number of runs in the D -optimal design. The optimal variance corresponding to D -optimal is calculated as $V(\emptyset_0) = (\text{trace } C_d)^{\frac{1}{S}}$. Where S is the number of parameters in the model.

2.1.1 Relative Efficiency of a Design

A measure of relative efficiency of design 1 to design 2 according to D-criterion is given as

$$D_e = \left[\frac{|(X_2' X_2)^{-1}|}{|(X_1' X_1)^{-1}|} \right]^{\frac{1}{S}} \quad (9)$$

where X_1 and X_2 are the X matrices for the two designs and S is the number of model parameters, [1]. Relative efficiency of the general to D-optimal design is equivalent to

$$\phi_{D\text{eff}(\xi)} = \frac{(\det C)^{\frac{1}{S}}}{(\det C_d)^{\frac{1}{S}}} \quad (10)$$

2.2 E-Optimal Efficiency

$$V(\emptyset_{-1}) = \left(\frac{1}{15} \text{trace } C_a^{-1}\right)^{-1} = \left(\frac{1}{15} (\sum_{j \leq N} \sqrt{b_{jj}})^2\right)^{-1} \quad (13)$$

obtain D -, A -, E - and T -optimal designs. The optimal value for the design is given by equation (7)

E-optimality is the minimization of the largest eigenvalue of $(X'X)^{-1}$, [13]. The minimum Eigen value of the moment matrix of the general design is $\lambda_{\min}(C)$ with a corresponding normalized eigenvector Z . If the minimum Eigen value have a multiplicity of one, the matrix $E = \frac{ZZ'}{\|Z\|^2}$ of trace one such that $x_i' E x_i \leq \lambda_{\min}(C)$ for all $x_i \in \chi$ for $\phi_{-\infty}$ -optimal is used to determine the E -optimal weights corresponding to the factorial and axial parts of the design respectively [12]. The optimal variance corresponding to the E -optimal design is the minimum Eigen value of the resulting moment matrix of the design i.e. $\lambda_{\min}(C_e)$ where $C_e = E'E/N$ and N is the number of rows of matrix E .

The E- efficiency of the general to the E -optimal design is

$$\phi_{E\text{eff}} = \frac{\lambda_{\min}(C)}{\lambda_{\min}(C_e)} \quad (11)$$

2.3. A-Optimal efficiency

The B_a matrix for computing the A -Optimal weight is obtained by substituting $p = -1$ in equation (6) so that

$$B_a = UU' \quad (12)$$

The A -Optimal factorial weight (A_{fw}) is $A_{fw} = \frac{\sqrt{b_{11}}}{\sum_{j \leq N} \sqrt{b_{jj}}}$ while the weight corresponding to axial part $A_{aw} = \frac{\sqrt{b_{NN}}}{\sum_{j \leq N} \sqrt{b_{jj}}}$ Where $b_{11}, \dots, b_{NN}$ are the diagonal elements of matrix B_a . The A -optimal design is obtained by replicating the factorial and axial parts as per the weights obtained above to give the design matrix X_a which is used to obtain the moment matrix $C_a = \frac{X_a' X_a}{N_a}$ where N_a is the number of runs in the A -Optimal design. The optimal variance corresponding to the A -optimal design is

The relative efficiency of the general design to the A –Optimal design is

$$\phi_{Aeff} = \frac{\left(\frac{1}{15} \text{trace } C^{-1}\right)^{-1}}{\left(\frac{1}{15} \text{trace } C_a^{-1}\right)^{-1}} \quad (14)$$

2.4. T –Optimal Efficiency

To obtain the weights corresponding to the T –optimal design, $p = 1$ in equation (6) and hence the B_t matrix is given as

$$B_t = UC^2U' \quad (15)$$

$$V(\phi_1) = \left(\frac{1}{s} \text{trace } C_t^1\right)^1 = \left(\frac{1}{s} \left(\sum_{j \leq N} \sqrt{b_{jj}}\right)^2\right)^1 \quad (16)$$

The T-efficiency for T-Optimal designs with the moment matrix C_t and the general design with moment matrix C is given as

$$\frac{\phi(C)}{\phi(C_t)} = \frac{\left\{\frac{1}{s} \text{trace } C\right\}}{\left\{\frac{1}{s} \text{trace } C_t\right\}} \quad (17)$$

Where s is the number of model parameters.

3.0 Results

3.1 D –Optimal Design and its Efficiency

Table 1 Moment Matrix for the D-Optimal Design.

$$M_d = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0109 & 0.0000 & 0.0000 & 0.0000 & 1.0109 & 0.0000 & 0.0000 & 1.0109 & 0.0000 & 1.0109 \\ 0.0000 & 1.0109 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0109 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0109 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0109 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0109 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 2.2809 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.6082 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0109 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.0000 & 0.0000 & 2.2809 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.6082 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0109 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.0000 & 2.2809 & 0.0000 & 0.6082 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 \\ 1.0109 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 0.0000 & 0.6082 & 0.0000 & 2.2809 \end{bmatrix}$$

The D-optimal value is $V(\phi_0) = (\det M_d)^{1/s}$. Hence the optimal variance is $V(\phi_0) = 0.6965612$, while the optimal value for the general design is 0.6796529 according to [9]. Therefore the relative efficiency of

Factorial weight corresponding to T –Optimal $T_{fw} = \frac{\sqrt{b_{11}}}{\sum_{j \leq N} \sqrt{b_{jj}}}$ while the weight corresponding to axial part is $T_{aw} = \frac{\sqrt{b_{NN}}}{\sum_{j \leq N} \sqrt{b_{jj}}}$. Where $b_{11}, \dots, b_{NN}$ are the diagonal elements of matrix B_t . The T –Optimal design is found by replicating the factorial and axial parts as per the weights obtained as above to give the design matrix X_t which is used to obtain the moment matrix $C_t = \frac{X_t' X_t}{N_t}$ where N_t is the number of runs in the T –Optimal design. The optimal variance corresponding to the T –Optimal design is

The weight corresponding to factorial part is 0.02387036 and that one corresponding to axial part is 0.02951858 as per equation (5). Hence a D –optimal design is a design with factorial part replicated approximately two times i.e. ($n_f = 2$) with the axial part being replicated three times (i.e. $n_a = 3$) giving a design matrix X with $32 \times 2 + 8 \times 3 = 88$ points with D –optimal moment matrix $M_d = \frac{1}{88} X'X$ given in table 1.

the general design to the D –optimal design is obtained

$$\phi_{eff(\xi)} = \frac{0.6796529}{0.6965612} = 97.71\% \cong 98\%.$$

3.2 E – Optimal Design and its Efficiency

$$Z' = [0.895 \ 0.00 \ 0.00 \ 0.00 \ 0.00 \ 0.00 \ -0.233 \ 0.00 \ 0.00 \ 0.00 \ -0.233 \ 0.00 \ 0.00 \ -0.233 \ 0.00 \ -0.233]$$

The E matrix $E = \frac{ZZ'}{\|Z\|^2}$ of trace one such that $x'Ex \leq \lambda_{\min}(MG)$ for all $x \in \chi$ for $\phi_{-\infty}$ -optimal $\forall x_i$ for $i = 1, \dots, 32$, $x \in \chi$, we have $x'Ex = 0.0009351244$ while for all x_i and $i = 33, \dots, 40$, $x'Ex = 0.01089277$ which is greater than the minimum

Table 2 Moment Matrix for the E-Optimal Design

$$M_e = \begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.970 & 0.000 & 0.000 & 0.000 & 0.970 & 0.000 & 0.000 & 0.970 & 0.000 & 0.970 \\ 0.000 & 0.970 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.970 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.970 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.970 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.970 & 0.000 & 0.000 & 0.000 & 0.000 & 1.255 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 0.836 & 0.000 & 0.836 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.970 & 0.000 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 0.000 & 1.255 & 0.000 & 0.000 & 0.836 & 0.000 & 0.836 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 0.000 \\ 0.970 & 0.000 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 1.255 & 0.000 & 0.836 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 \\ 0.970 & 0.000 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 0.000 & 0.836 & 0.000 & 0.000 & 0.836 & 0.000 & 1.255 \end{bmatrix}$$

With an optimal variance of 0.4182000. The relative efficiency of the general to the $\phi_{-\infty}$ -optimal design is obtained using equation (11) which is 1%.

3.3 A – Optimal Design and its Efficiency

The A -optimal weights corresponding to factorial and axial parts are 0.01704711 and 0.05681391

The minimum Eigen value of the moment matrix of the general design is $\lambda_{\min}(MG) = 0.002893284$ with a corresponding normalized Eigenvector Z given as

eigenvalue therefore the E -optimal design assigns a weight of one to the factorial part and zero to the axial part of the design. The moment matrix of the $\phi_{-\infty}$ -optimal design is $M_e = \frac{1}{32}X'X$ tab

respectively. Therefore the A -optimal design is obtained by replicating the factorial part two times (i.e. $n_f = 2$) and the axial part six times (i.e. $n_a = 6$) giving a design with $32 \times 2 + 8 \times 6 = 112$ and A -optimal moment matrix is $M_A = \frac{1}{112}X'X$

Table 3: Moment Matrix for A-Optimal Design

$$M_A = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0343 & 0.0000 & 0.0000 & 0.0000 & 1.0343 & 0.0000 & 0.0000 & 1.0343 & 0.0000 & 1.0343 \\ 0.0000 & 1.0343 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0343 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0343 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0343 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0343 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 2.8674 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.4779 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0343 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 0.0000 & 2.8674 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.4779 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 0.0000 \\ 1.0343 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 2.8674 & 0.0000 & 0.4779 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 \\ 1.0343 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 0.0000 & 0.4779 & 0.0000 & 2.8674 \end{bmatrix}$$

The A-Optimal value for optimal design is $V_{-1} = \left(\frac{1}{15} \text{trace} M_a^{-1}\right)^{-1}$ is found to be 0.05798174 while the optimal value for the general design was 0.04154701. The relative efficiency of the general compared to the optimal is $0.71655334938 \cong 71.7\%$.

3.4 T-Optimal Design and its Efficiency

While the T -optimal weights corresponding to factorial and axial parts were 0.01691205 and 0.05735179 respectively as per equations (5) when $p = 1$ in equation (6). Again the T -optimal design was approximated by replicating the factorial part of the general design two times and the axial part six times giving a design with a total of $32 \times 2 +$

$8 \times 6 = 112$ runs .The T -optimal design moment matrix is the same as the one in table 3. The T -optimal value $V_1 = \frac{1}{15} \text{trace} MG = 1.29828$. Relative efficiency of the general design to the T -optimal design is $\frac{1.136227}{1.29828} \cong 87.5\%$.

4.0 Conclusions

The E-optimal design is the most efficient design with a relative efficiency of 1% to the general design with the factorial part replicated once this design has the least number of runs 32 only. Hence to model a process with four in put variables using a second order rotatable design constructed using BIBD when the replications are less than three the number of times pairs of treatments occur together in the design, this design is the most efficient saving time and costs due to the least number of runs.

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Modeling Tuberculosis Treatment Outcomes using a Discrete Time Markov Chain Model

Saidi J. K. and Musiga L. A.

School of Mathematics, University of Nairobi

P.O. Box 30197 - 00100, Nairobi, Kenya

E-mails: saidijames33@gmail.com, lmusiga@uonbi.ac.ke

Abstract

Tuberculosis (TB) is an air-borne disease which can be fatal if not properly managed. Upon diagnosis, treatment regimens are usually instituted thus resulting in various treatment outcomes whose prevalence rates are of major interest. In this study, the various TB treatment outcomes were modelled using the theory of Markov Chains, specifically the Discrete Time Markov Chain model. Furthermore, the Chapman-Kolmogorov result was used to predict the future TB treatment outcomes prevalence rates with a view to projections for mitigation. The data for this study was obtained from the Kenyan Ministry of Health and octave software was used for analysis. It was found that for the patients who were detected with TB, 66.78% got cured, 7.06% got lost to follow-up, 0.95% were moved to category 4 treatment while 5.94% died. On the other hand, for patients who were not detected with TB, 2.04% got cured, 2.38% got lost to follow-up, 0.68% were moved to category 4 treatment while 12.24% died. Additionally, out of the patients who were Rifampicin Resistant and patients who had Extra-Pulmonary TB, only 47.54% and 3.13% got cured, respectively.

Keywords: Disease, Detection, Management, Treatment Regimens

1 Introduction

Tuberculosis is a disease that affects the lungs and other body organs and is usually spread from one person to another through coughing, sneezing and spitting. Since Tuberculosis is an air-borne disease, its' infection rates are high and it can be fatal when not managed properly. Various studies which have been conducted on the disease are presented as follows.

Hoad *et al.* (2009) used a stochastic Markov chain model to find out the effects of local and global parameters on the possibility of contracting Tuberculosis in Gem and Asembo Divisions of Western Kenya. Variations of the model identified how the local effect depended on factors such as age and gender. Geographical clusters of people were divided into two groups and the effectiveness of an intervention policy

was assessed by applying it to one group but not the other. The results indicated that the local transmission coefficient was higher than the global transmission coefficient. Ozcaglar *et al.* (2012) reviewed earlier studies on modeling different aspects of Tuberculosis dynamics. The models simulated Tuberculosis transmission dynamics, treatment, drug resistance, control strategies for increasing compliance to treatment, HIV/TB co-infection and patient groups. Ordinary differential equations, simulation models and Markov Chain Monte Carlo methods were used. They noted that epidemiological models gave an overall view of Tuberculosis dynamics and directed efforts for surveillance and elimination of Tuberculosis. In a study conducted by Hill *et al.* (2012) a system of differential equations were used to model Tuberculosis in the USA. They evaluated the intervention strategies on time to elimination, treatment of Tuberculosis active cases, the treatment of inactive infection and reduction of foreign-born persons who entered the country with latent Tuberculosis infection. The study predicted that it would take more than 20 years to achieve elimination among the US-born population. In Kenya, Kosgei *et al.* (2015) used multivariate and logistic regression to explain the impact of gender difference towards the treatment outcomes for 15 to 49 year-old persons diagnosed with Pulmonary Tuberculosis. Furthermore, they investigated the factors linked to poor treatment outcomes. Their results indicated that more females than males got a poor treatment outcome, this being attributed to cultural inequalities and socio-economic differences. Xu *et al.* (2017) used an eight-state Markov Chain model to predict the Tuberculosis epidemic over 50 years among the general population in China. They determined the influencing factors and suggested a specific 90-90-90 strategy for Tuberculosis under which the prevalence would be 1.77 cases per 100 000 population within 50 years. Undetected smear-negative Tuberculosis formed the majority of the infected patients and with the prevailing interventions, they projected that Tuberculosis prevalence would stagnate at 163 cases per 100,000 after 50 years. Onyango *et al.* (2018) analysed Tuberculosis data for 23,753 children who were less than 15 years in age, extracted from the Kenyan National Tuberculosis Program. Survival analysis was performed with censorship at 9 months and mortality was the main outcome. Cox proportional hazards regression was used for assessing the risk factors for mortality which were identified as being HIV infected but not on ARV therapy, being HIV infected and on ARV therapy, being a child with Tuberculosis below the age of 5 years and being diagnosed with Pulmonary disease. It was found that childhood Tuberculosis accounted for 9% of all patients, HIV/TB co-infection rate was 28%, treatment was successful in 90% but 4% died. In another study, Li *et al.* (2020) used dynamic Markov modelling with eight states to predict the longitudinal trends of Multi-Drug-Resistant Tuberculosis (MDR-TB) in China, by 2050. The prevalence of MDR-TB and Drug-Susceptible Tuberculosis (DS-TB+) were simulated and MDR-TB was stratified into whether the disease was treated with the recommended regimen or not. The authors found that without any changes in intervention, the prevalence of DS-TB+ was projected to decline to 20 per

100 000 people, whereas that of MDR-TB was expected to triple to 58 per 100 000 people by 2050. Adeboye et al. (2020) modelled a joint generalized logit model for binary variables and Cox proportional hazards of latent class for survival times, which was applied to Tuberculosis data obtained from Eastern Cape Province, South Africa. A Bayesian approach for maximum likelihood estimation was implemented using Markov Chain Monte Carlo technique. The results indicated that age, gender, smoking and diabetes were significant covariates. The joint model provided a better predictive power for Tuberculosis prognostic factors.

2 The Discrete Time Markov Chain Model

2.1 The Initial Transition Probability Matrix

Consider a Markov Chain model with s non-absorbing states, $s = 1, 2, 3, \dots, s$ and r absorbing states, $r = 1, 2, 3, \dots, r$. Then the total number of possible states of the system are $r + s = N$. The transition probability matrix P of the Markov Chain is represented in the following canonical form, assuming time homogeneity;

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ r_{11} & r_{12} & r_{13} & \cdots & r_{1r} & q_{11} & q_{12} & q_{13} & \cdots & q_{1s} \\ r_{21} & r_{22} & r_{23} & \cdots & r_{2r} & q_{21} & q_{22} & q_{23} & \cdots & q_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{s1} & r_{s2} & r_{s3} & \cdots & r_{sr} & q_{s1} & q_{s2} & q_{s3} & \cdots & q_{ss} \end{pmatrix} \quad (1)$$

which is compactly represented as

$$P = \begin{pmatrix} I & O \\ R & Q \end{pmatrix} \quad (2)$$

where

I is an $r \times r$ identity matrix which gives the transition probabilities between absorbing states.

O is an $r \times s$ matrix of zeroes which gives the transition probabilities from absorbing to non-absorbing states.

$Q = ((q_{ij}))$ is an $s \times s$ matrix, q_{ij} being the probability of moving from state i at time $(t - 1)$ to state j at time t , $i, j = 1, 2, 3, \dots, s$.

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$R = ((r_{ik}))$ is an $s \times r$ matrix, r_{ik} being the probability of moving from state i at time $(t - 1)$ to state k at time t , $i = 1, 2, \dots, s$ and $k = 1, 2, \dots, r$.

2.2 The n-Step Transition Probability Matrix

According to the Chapman-Kolmogorov result, the n-step transition probability matrix is given by

$$P^n = \begin{pmatrix} I & O \\ (I + Q + Q^2 + Q^3 + \dots + Q^{n-1})R & Q^n \end{pmatrix} \quad (3)$$

that is

$$P^n = \begin{pmatrix} I & O \\ R^n & Q^n \end{pmatrix} \quad (4)$$

where

I is an $r \times r$ identity matrix which gives the transition probabilities between absorbing states in n-steps.

O is an $r \times s$ matrix of zeroes which gives the transition probabilities from absorbing states to non-absorbing states in n-steps.

$R^n = ((r_{ik}^{(n)})) = (I + Q + Q^2 + Q^3 + \dots + Q^{n-1})R$ is an $s \times r$ matrix which gives the transition probability of moving from state i at time $(t - 1)$ to absorbing state k at time t in n-steps, $i = 1, 2, \dots, s$ and $k = 1, 2, \dots, r$.

$Q^n = ((q_{ij}^{(n)}))$ is an $s \times s$ matrix which gives the transition probability of moving from state i at time $(t - 1)$ to state j at time t in n-steps, $i, j = 1, 2, 3, \dots, s$.

In the long run

$$\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} I & O \\ (I - Q)^{-1}R & O \end{pmatrix} \quad (5)$$

The matrix $(I - Q)^{-1}$ is the Fundamental matrix of the Absorbing Markov Chain whereas the matrix $(I - Q)^{-1}R$, which is the Limiting matrix of the Absorbing Markov Chain, gives the long run probabilities of moving from the non-absorbing states to the absorbing states.

3 An Application

The management of Tuberculosis patients involves being put on various treatment regimens accompanied with regular follow-ups. The possible treatment outcomes are being cured, dying, getting lost to follow-up, moving to treatment 4 category, not completing the treatment and being transferred to other treatment facilities. Thus, starting off with a cohort of Tuberculosis patients, the movements or transitions of the patients into the various treatment outcomes in a specified time interval can easily be obtained. This behaviour is well captured using a Discrete Time Markov Chain model which comprises of transitions between different states considered over a specified time interval, the states in this study being the various treatment outcomes for which transitions are considered over a one year period. The Tuberculosis data for this study was obtained from the Kenyan Ministry of Health. The data captured 1456 patients who were on treatment in various government hospitals across the country for the period between January 2017 and December 2017. The states that were categorized as absorbing states were Cured(C) and Dead(D) while the states that were considered as non-absorbing states comprised of MTB Detected(MD), MTB Not Detected(MND), Rifampicin Resistant(RR), Not Rifampicin Resistant(NRR), Pulmonary Tuberculosis(P), Extra-Pulmonary Tuberculosis(EP), Treatment Failure(F), Lost To Follow Up(LTFU), Moved To Category 4 Treatment(MT4), Not Completed Treatment(NC), Completed Treatment(TC) and Transferred Out(TO). The Tuberculosis data for the study is shown in Table 3.1.

Table 3.1. Tuberculosis Data 2017

	Cured	Dead	F	LTFU	MT4	NC	TC	TO	Total
MTB-Detected (MD)	776	69	7	82	11	2	185	30	1162
MTB-Not Detected (MND)	6	36	0	7	2	0	237	6	294
Rifampicin Resistant (RR)	29	4	1	4	2	0	21	0	61
Not Rifampicin Resistant (NRR)	753	101	6	85	11	2	401	36	1395
Pulmonary TB (P)	780	98	7	88	12	2	371	34	1392
Extra-Pulmonary TB (EP)	2	7	0	1	1	0	51	2	64

3.1 Initial Transition Probability Matrix

Let the states of the Tuberculosis patients be denoted by the integers $1, 2, \dots, N$ at times $t = 1, 2, \dots$. Let p_{ij} denote the probability that a patient in state i at time $(t - 1)$ will be in state j at time t hence the transition matrix $P = ((p_{ij})), i, j = 1, 2, \dots, N$. Let $n_{ij}(t)$ represent the number of patients in state i at time $(t - 1)$ who transit to state j at time t . Also let $n_i(t - 1)$ represent the number of patients in state i at time $(t - 1)$, then the transition probabilities are estimated using

$$p_{ij} = \frac{n_{ij}(t)}{n_i(t-1)} \quad (6)$$

where $i, j = 1, 2, \dots, N$. This is the proportion of patients who were in state i at time $(t-1)$ who transit to state j at time t .

Thus the one-step transition probability for the patients who were MTB detected and got cured is obtained as $776/1162=0.6678$ while the one-step transition probability for the patients who were MTB detected and completed their treatment is $185/1162 = 0.1592$. Hence assuming time homogeneity, the one-step transition probability matrix P is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6678 & 0.0594 & 0.0060 & 0.0706 & 0.0095 & 0.0017 & 0.1592 & 0.0258 \\ 0.0204 & 0.1224 & 0 & 0.0238 & 0.0068 & 0 & 0.8062 & 0.0204 \\ 0.4754 & 0.0656 & 0.0164 & 0.0656 & 0.0327 & 0 & 0.3443 & 0 \\ 0.5398 & 0.0724 & 0.0043 & 0.0609 & 0.0079 & 0.0014 & 0.2875 & 0.0258 \\ 0.5605 & 0.0704 & 0.0050 & 0.0632 & 0.0086 & 0.0014 & 0.2665 & 0.0244 \\ 0.0313 & 0.1094 & 0 & 0.0156 & 0.0156 & 0 & 0.7968 & 0.0313 \end{pmatrix} \quad (7)$$

The initial transition matrix then obtains the n-step transition probabilities which are presented and discussed in the following subsection.

3.2 n-Step Transition Probabilities and Discussions

Considering the patients who were Detected with Tuberculosis (MD), the n-step transition probabilities are obtained as shown in Table 3.2.

Table 3.2. n-Step Transition Probabilities for patients who were MTB Detected

$P^{(n)}$	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	$P^{(4)}$	$P^{(5)}$	$P^{(6)}$	$P^{(7)}$	$P^{(8)}$	$P^{(9)}$	$P^{(10)}$
C	0.6678	0.7687	0.8411	0.8706	0.8845	0.8908	0.8937	0.8950	0.8956	0.8959
D	0.0594	0.0832	0.0945	0.0996	0.1020	0.1030	0.1035	0.1037	0.1038	0.1039
F	0.0060	0.0010	0.0007	0.0003	0.0001	0.0001	0	0	0	0
LTFU	0.0706	0.0133	0.0085	0.0036	0.0017	0.0008	0.0003	0.0002	0.0001	0
MT4	0.0095	0.0026	0.0014	0.0006	0.0003	0.0001	0.0001	0	0	0
NC	0.0017	0.0002	0.0002	0.0001	0	0	0	0	0	0
TC	0.1592	0.1246	0.0501	0.0237	0.0107	0.0049	0.0022	0.0010	0.0005	0.0002
TO	0.0258	0.0063	0.0035	0.0015	0.0007	0.0003	0.0001	0.0001	0	0

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According to Table 3.2, 66.78% of the patients got cured, 5.94% died, 7.06% got lost to follow-up while 0.95% were moved to category 4 treatment. It is projected that in eight years' time, 89.5% of the patients will get cured, 10.37% will die, 0.02% will get lost to follow up and none will be moved to category 4 treatment.

Secondly, the n-step transition probabilities for the category of patients who were Not Detected with Tuberculosis (MND) are displayed in Table 3.3.

Table 3.3. n-Step Transition Probabilities for patients who were MTB Not Detected

$P^{(n)}$	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	$P^{(4)}$	$P^{(5)}$	$P^{(6)}$	$P^{(7)}$	$P^{(8)}$	$P^{(9)}$	$P^{(10)}$
C	0.0204	0.4766	0.6270	0.7029	0.7365	0.7520	0.7590	0.7622	0.7637	0.7643
D	0.1224	0.1847	0.2120	0.2246	0.2303	0.2329	0.2341	0.2347	0.2349	0.2350
F	0	0.0041	0.0014	0.0007	0.0003	0.0001	0.0001	0	0	0
LTFU	0.0238	0.0523	0.0184	0.0091	0.0040	0.0019	0.0008	0.0004	0.0002	0.0001
MT4	0.0068	0.0076	0.0032	0.0015	0.0007	0.0003	0.0001	0.0001	0	0
NC	0	0.0011	0.0004	0.0002	0.0001	0	0	0	0	0
TC	0.8062	0.2526	0.1297	0.0572	0.0263	0.0120	0.0054	0.0025	0.0011	0.0005
TO	0.0204	0.0208	0.0080	0.0038	0.0017	0.0008	0.0004	0.0002	0.0001	0

Looking at Table 3.3, a paltry 2.04% of the patients got cured, 12.24% died, 2.38% got lost to follow-up while 0.68% were moved to category 4 treatment. It is projected that in ten years' time, 76.43% of the patients will get cured, 23.50% will die, 0.01% will get lost to follow up and none will be moved to category 4 treatment.

Third is the category of patients who were Rifampicin Resistant (RR), for whom the n-step transition probabilities obtained are shown in Table 3.4.

Table 3.4. n-Step Transition Probabilities for patients who were Rifampicin Resistant

$P^{(n)}$	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	$P^{(4)}$	$P^{(5)}$	$P^{(6)}$	$P^{(7)}$	$P^{(8)}$	$P^{(9)}$	$P^{(10)}$
C	0.4754	0.6962	0.7900	0.8334	0.8531	0.8621	0.8662	0.8680	0.8689	0.8693
D	0.0656	0.1010	0.1170	0.1243	0.1276	0.1291	0.1298	0.1302	0.1303	0.1304
F	0.0164	0.0024	0.0009	0.0004	0.0002	0.0001	0	0	0	0
LTFU	0.0656	0.0266	0.0113	0.0052	0.0024	0.0011	0.0005	0.0002	0.0001	0
MT4	0.0327	0.0046	0.0019	0.0009	0.0004	0.0002	0.0001	0	0	0
NC	0	0.0005	0.0002	0.0001	0	0	0	0	0	0
TC	0.3443	0.1585	0.0739	0.0335	0.0153	0.0070	0.0032	0.0014	0.0007	0.0003
TO	0	0.0102	0.0048	0.0022	0.0010	0.0005	0.0002	0.0001	0	0

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Table 3.4 shows that of this category, 47.54% of patients got cured, 6.56% died, 6.56% got lost to follow-up while 3.27% were moved to category 4 treatment. It is projected that in five years' time, 85.31% of the patients will get cured, 12.76% will die, 0.24% will get lost to follow up and 0.04% will be moved to category 4 treatment.

The next category is that of the patients who were Not Rifampicin Resistant (NRR). The n-step transition probabilities obtained for this category are displayed in Table 3.5.

Table 3.5 n-Step Transition Probabilities for patients who were Not Rifampicin Resistant

$P^{(n)}$	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	$P^{(4)}$	$P^{(5)}$	$P^{(6)}$	$P^{(7)}$	$P^{(8)}$	$P^{(9)}$	$P^{(10)}$
C	0.5398	0.7104	0.7982	0.8369	0.8547	0.8628	0.8665	0.8682	0.8690	0.8693
D	0.0724	0.1038	0.1183	0.1249	0.1279	0.1292	0.1299	0.1301	0.1303	0.1303
F	0.0043	0.0016	0.0008	0.0004	0.0002	0.0001	0	0	0	0
LTFU	0.0609	0.0209	0.0105	0.0047	0.0021	0.0010	0.0004	0.0002	0.0001	0
MT4	0.0079	0.0036	0.0017	0.0008	0.0004	0.0002	0.0001	0	0	0
NC	0.0014	0.0004	0.0002	0.0001	0	0	0	0	0	0
TC	0.2875	0.1501	0.0658	0.0303	0.0138	0.0063	0.0029	0.0013	0.0006	0.0003
TO	0.0258	0.0092	0.0044	0.0020	0.0009	0.0004	0.0002	0.0001	0	0

As per Table 3.5, 53.98% of the patients who were Not Rifampicin Resistant got cured, 7.24% died, 6.09% got lost to follow-up while 0.79% were moved to category 4 treatment. It is projected that in six years' time, 86.28% of the patients will get cured, 12.92% will die, 0.10% will get lost to follow up and 0.02% will be moved to category 4 treatment.

Table 3.6 below displays the n-step transition probabilities obtained for the category of the patients who had Pulmonary Tuberculosis (P).

Table 3.6. n-Step Transition Probabilities for patients who had Pulmonary Tuberculosis

$P^{(n)}$	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	$P^{(4)}$	$P^{(5)}$	$P^{(6)}$	$P^{(7)}$	$P^{(8)}$	$P^{(9)}$	$P^{(10)}$
C	0.5605	0.7201	0.8052	0.8424	0.8596	0.8674	0.8709	0.8725	0.8733	0.8736
D	0.0704	0.1005	0.1145	0.1208	0.1237	0.1250	0.1256	0.1259	0.1260	0.1261
F	0.0050	0.0015	0.0008	0.0003	0.0002	0.0001	0	0	0	0
LTFU	0.0632	0.0197	0.0102	0.0045	0.0021	0.0009	0.0004	0.0002	0.0001	0
MT4	0.0086	0.0034	0.0017	0.0007	0.0003	0.0002	0.0001	0	0	0
NC	0.0014	0.0004	0.0002	0.0001	0	0	0	0	0	0
TC	0.2665	0.1456	0.0632	0.0292	0.0132	0.0060	0.0028	0.0013	0.0006	0.0003
TO	0.0244	0.0087	0.0043	0.0019	0.0009	0.0004	0.0002	0.0001	0	0

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According to Table 3.6, 56.05% of the patients got cured, 7.04% died, 6.32% got lost to follow-up while 0.86% were moved to category 4 treatment. It is projected that in seven years' time, 87.09% of the patients will get cured, 12.56% will die, 0.04% will get lost to follow up and 0.01% will be moved to category 4 treatment.

The last category is made up of the group of patients who had Extra-Pulmonary Tuberculosis (EP). The n-step transition probabilities are displayed in Table 3.7.

Table 3.7 n-Step Transition Probabilities for patients who had Extra-Pulmonary Tuberculosis

$P^{(n)}$	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	$P^{(4)}$	$P^{(5)}$	$P^{(6)}$	$P^{(7)}$	$P^{(8)}$	$P^{(9)}$	$P^{(10)}$
C	0.0313	0.4866	0.6386	0.7150	0.7489	0.7644	0.7715	0.7747	0.7762	0.7769
D	0.1094	0.1719	0.1993	0.2120	0.2177	0.2204	0.2216	0.2221	0.2224	0.2225
F	0	0.0042	0.0014	0.0007	0.0003	0.0001	0.0001	0	0	0
LTFU	0.0156	0.0522	0.0186	0.0091	0.0041	0.0019	0.0009	0.0004	0.0002	0.0001
MT4	0.0156	0.0080	0.0032	0.0015	0.0007	0.0003	0.0001	0.0001	0	0
NC	0	0.0011	0.0004	0.0002	0.0001	0	0	0	0	0
TC	0.7968	0.2552	0.1304	0.0576	0.0265	0.0120	0.0055	0.0025	0.0011	0.0005
TO	0.0313	0.0207	0.0081	0.0039	0.0017	0.0008	0.0004	0.0002	0.0001	0

Looking at Table 3.7, only 3.13% of the patients got cured, 10.94% died, 1.56% got lost to follow-up while 1.56% were moved to category 4 treatment. It is projected that in two years' time, 48.66% of the patients will get cured, 17.19% will die, 5.22% will get lost to follow up and 0.80% will be moved to category 4 treatment.

Furthermore, using the Limiting matrix of the Absorbing Markov Chain, the long run transition probabilities of moving from the non-absorbing states to the absorbing states were obtained as shown in equation (8).

$$\begin{pmatrix} 0.8961 & 0.1039 \\ 0.7649 & 0.2351 \\ 0.8696 & 0.1304 \\ 0.8696 & 0.1304 \\ 0.8739 & 0.1261 \\ 0.7774 & 0.2226 \end{pmatrix} \quad (8)$$

This means that in the long run, it is expected that 89.61% of patients Detected with Tuberculosis will get cured while 10.39% will die. On the other hand, 76.49% of patients Not Detected with Tuberculosis are expected to get cured while 23.51% are expected to die. Considering Rifampicin Resistant and

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Not Rifampicin Resistant patients, 86.96% are expected to get cured while 13.04% are expected to die, respectively. Of the patients with Pulmonary Tuberculosis, 87.39% are expected to get cured while 12.61% are expected to die. Last but not least, 77.74% of patients with Extra-Pulmonary Tuberculosis are expected to be cured while 22.26% are expected to die.

5 Conclusions

In this paper, Tuberculosis treatment outcomes have been modelled using a Discrete Time Markov Chain model. The chances of arriving at any of the treatment outcomes after diagnosis and management have been ascertained. Predictions of the future outlook as well as the long run scenario have been made in order for planning for mitigation measures against the disease.

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EVALUATION OF THE PRESENCE OF RANDOM EFFECTS ON MODELING AND OPTIMIZING CULTURE CONDITIONS OF MILK KEFIR GRAINS USING RESPONSE SURFACE METHODOLOGY.

Edwine B. Atitwa ^{a*}, Joseph.K.Koske author ^b, John M.Muindi ^c

^a Department of Computer and Statistics, Moi University,
P.O. Box 3900, Eldoret, and 30100, Kenya, Address,

^b Department of Computer and Statistics, Moi University,
P.O. Box 3900, Eldoret, and 30100, Kenya

^c Department of Computer and Statistics, Moi University,
P.O. Box 3900, Eldoret, and 30100, Kenya

^a Email: eatitwa@gmail.com

^b Email: Koske4@yahoo.co.uk

^c Email: johnkasome@yahoo.com

Abstract

This study provides evaluation of presence of random effects on the growth of kefir grains. Culture conditions were optimized and modelled to determine the growth of kefir grains using milk. The study aims at determining presence of random effects that may interfere growth of kefir grains using milk. The operational conditions considered during the growth of the kefir grains were number of rotations, fat content and time. This led to an increase in biomass of kefir grains with high level of probiotics which are nutritious and healthy. In order to establish presence of random effects during optimum growth of kefir grains; culture conditions, culture liquid and presence of random effects were considered. Thus, an experiment was carried out in order to obtain the data which was used to establish the necessary conditions on the growth of kefir grains. The Design expert was used to extract data through controlled experiment. Box-Behnken design was used to model the data. Evaluation of random effects to the growth of kefir grains was tested by variance analysis through blocking. The variance, covariance and standard errors were computed to assess the presence of random effects in each block. The tests for random effects were insensitive and remainder sum of squares was not uniform across the blocks hence presence of random effects. The study indicated the presence of random effects which should be controlled during growth.

Keywords: Random effects, blocks, standard error, kefir grains, box-behnken design, and probiotics.

1.0 Introduction

1.1 Background information

Kefir grains are combinations of yeasts and bacteria living on a substrate made up of a variety of dairy components (Lewis, 2012). Furthermore, probiotic bacteria in kefir grains have shown many health-promoting properties, such as inhibition of detrimental effect of pathogens, enhancement of the intestinal barrier and modulation of the immune response. These health benefits have been observed both *in vitro* and *in vivo* experiments (Courtesy et al., 2007). Through the magic of fermentation, kefir grains adds more vitamins and special properties to an ordinary substance, healing so many who consume it (Dominic et al. ,2015). Kefir grains can be growth using various culture liquids including milk as seen in this study. Kefir is a fantastic source of other nutrients, it lowers blood pressure, lowers blood sugar, good for cancer patients especially when going through chemotherapy, it can help

people with allergies, helps the detoxification process by binding to some mutagens, agents that can literally change your DNA, they are wonderful for digestion, it is a whole food for nutrients and health (Dominic et al., 2015).

When treatments are from a continuous range of values, then a Response Surface Methodology (RSM) is used since modeling and analysis of problems in which a response of interest is influenced by several variables was included and the objective was to optimize this response (Montgomery, 2005).

The application of RSM to design optimization is aimed at reducing the cost of expensive analysis methods (e.g. finite element method or Complete Factorial Design (CFD) analysis) and their associated numerical noises (Venter, 1998).

The Box-Behnken Design (BBD) was used to determine the runs that were used for data

processing during growth of kefir grains using milk. Optimal level of culture conditions at rotations being high, time being low level and fat content being at high level (Atitwa et al., 2016).

1.2 Problem statement

Considering the diversity in composition of the natural biomass in kefir grain production, as well as the structure and physiochemical properties of these probiotics, a universal processing protocol is not conceivable, and specific processes must be designed and optimized for each growth size depending on different factors and the culture composition used (Molina et al., 2013). For such

1.3 Purpose of the study

This study was done to evaluate the presence of random effects during grow of milk kefir grains. In this work, random effects are eliminated so that the milk kefir grain to be grown maintains high yield of probiotics

Hence through process should be carried out to avoid presence of random effects during growth of kefir grains to avoid interfering with its probiotic structures during its preparation.

programmes to be effective, it is essential that the process is accurate and this will be so when culture conditions are well monitored during preparation of kefir grains using milk (Atitwa et al., 2016). Thus the problem of this research to find out if random effects exists during growth of kefir grains considering given levels of culture conditions (Number of rotations, time and fat content) used.

which is good for nutrition and health for human life.

This will result to kefir grains that can be of importance especially for cancer patients when undergoing chemotherapy and to play very vital role in human life.

2.0 Literature Review

Perni et al.(2005) found out that the statistical indicators proved that the Grade of membership model was the best for describing the kefir grain biomass growth curve during traditional batch propagation in milk. He thus concluded that the results will always depend on specific selected bioprocess conditions, propagation medium and for the initial source of kefir grain culture. From this study it was well known that microbial composition in milk and kefir grains considerably vary with time and are dependent on age and storage conditions. Jeo Gao et al. (2012) used response surface analysis to optimize Tibetan biomass production. Results showed that optimal culture conditions were skim milk concentration, 41.6%; temperature 30.05°C; inoculation amount, 1.86%; time, 20h and

shaker rotating speed, 0r/min, at which the growth rate was 14.33%, 39.4% more than initial. He therefore concluded that using addition skim milk concentration, temperature and inoculation amount increases biomass production of kefir grains.

Thus all these experiments indicates that the growth of kefir grains using milk is associated with number of factors which need not be eliminated or controlled during preparation(Atitwa et al.,2016).

According to Richard (2011), blocking can be used to group the runs in to uniform blocks hence eliminating presence of block effects. This is done to isolate a systematic effect and prevents it from obscuring the main effects.

3.0 Material and Methods

3.1 Study area

An experiment was conducted within a period of 3 months where 27 independent experiments were carried out and repeated 3 times to determine the accuracy where results of each run were averaged. The material used were dairy milk which was culture liquid and culture conditions were time, fat content and number of rotations. The experiment was conducted locally in a kitchen at room temperature. Where, in addition to the main control factors, the response may be subject to variations due to the presence of some random effects (Khuri, 2010). For example, the type of milk used to perform the experiment may be selected at random from the supplier. This may differ in quality, thus the response model can include a random effect to account for the milk to-milk variability or the origin of kefir grain.

3.2 Construction of blocks

Random effects were evaluated by first grouping the runs in to blocks which were developed by confounding the runs in different groups by use of general definition of contrast which was given as

$$L = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_q x_q \quad (3.1)$$

Where α_i represents the exponents on the i^{th} factor in the effect to be confounded and x_i is the level of the i^{th} factor in a particular treatment combination. Thus, x_i takes the values of -1 (low level), 0 (intermediate level), or 1 (high level), where α_i is -1, 0, or 1. The runs were assigned in μ/h blocks.

Calculation of the estimates $b_0, b_1, b_{ii},$ and b_{ij} for each factor in the design and then to add further columns corresponding to $x_1^2, x_2^2, \dots, x_k^2, x_1 x_2, x_1 x_3, \dots, x_{k-1} x_k$. According to Box and Behnken (1960) estimates were computed as follows

$$i) \{iy\} = \sum_{u=1}^N X_{iu} Y_u, \quad ii) \{iijy\} = \sum_{u=1}^N X_{iu}^2 Y_u, \quad iii) \{ijjy\} = \sum_{u=1}^N X_{iu} X_{ju} Y_u \quad (3.2)$$

$$\left. \begin{aligned} b_0 &= \bar{y} \\ b_i &= A \{iy\} \\ b_{ii} &= B \{iy\} + C_1 \sum_i^{n_1} \{jjy\} + C_2 \sum_i^{n_2} \{lly\} - \left(\frac{\bar{y}_0}{s} \right) \end{aligned} \right\} 3.3$$

Where

$\sum_i^{n_1}$ and $\sum_i^{n_2}$ refer to summation over first

and second associates of i and

$$\begin{aligned} b_{ij} &= D_1 \{ijy\} & \text{i,j first associates} \\ b_{ij} &= D_2 \{ijy\} & \text{i,j second associates} \end{aligned}$$

The analysis of variance table was

calculated as follows using the following

formulas:

$$\text{Correction due to the means: } \frac{\{Oy\}^2}{N} \quad (3.4)$$

Sum of squares due to linear terms:

$$A \sum_{i=1}^k \{iy\}^2 \quad (3.5)$$

a) Due to interaction terms:

$$D_1 \sum_{i < j}^{n_1} \{ijy\}^2 + D_2 \sum_{i < j}^{n_2} \{ijy\}^2 \quad (3.6)$$

b) Due to quadratic terms :

$$b_0 \{Oy\} + \sum_{i=1}^k b_{ii} \{iijy\} - \frac{\{Oy\}^2}{N} \quad (3.7)$$

Total sum of squares after correction for the

$$\text{mean: } \sum_{u=1}^N y_u^2 - \frac{\{Oy\}^2}{N} \quad (3.8)$$

And the variance and covariance of the

estimates were determined as follows

$$\left. \begin{aligned} V(b_0) &= \frac{1}{n_0} \sigma^2 \\ V(b_i) &= A \sigma^2 \\ V(b_{ii}) &= [B + 1 / s^2 n_0] \sigma^2 \\ V(b_{ij}) &= D_1 \sigma^2 & \text{i,j first associates.} \\ &= D_2 \sigma^2 & \text{i,j second associates.} \\ \text{Cov}(b_0, b_{ii}) &= -\frac{1}{s^2 n_0} \sigma^2 \\ \text{Cov}(b_{ii}, b_{ij}) &= \left[C_1 + \frac{1}{s^2 n_0} \right] \sigma^2, & \text{i,j first associates.} \\ &= \left[C_2 + \frac{1}{s^2 n_0} \right] \sigma^2, & \text{i,j second associates} \end{aligned} \right\} (3.9)$$

4.0 Results

This section displays results, interpretation and discussion of findings as follows

4.1 Confounding the box-behnken design

Design (3.2) was confounded 3 blocks of 5 runs in each block. This experiment can be placed in the blocks of unit, where units in the same block are homogeneous. First, it was necessary to define a contrast by choosing a factorial effect to confound with blocks. For this study the contrast was defined using (3.1) as follows,

For example let AB and AC be the two components of interaction chosen to construct the design. The two defining contrasts for assigning runs to blocks were;

$$L_1 = xA + xB = u \quad (\text{Mod } 2)$$

$$u = -1, 0, 1$$

$$L_2 = xA + xC = h \quad (\text{Mod } 2)$$

$$h = -1, 0, 1$$

The L equations were to take only the values of -1, 0, or 1 because of $L \pmod{2}$. As a result, the treatment combinations in the 3^3 design assigned to the blocks based on the values of u and h , denoted as u/h block. For example, the -1,-1, 0 treatment combination had an u value of

$$u = 1(-1) + 1(-1) + 0(0) = -2 \pmod{2} = 0$$

and had an h value of

$$h = 1(-1) + 0(-1) + 1(0) = -1 \pmod{2} = 1$$

Therefore, the treatment combination -1,-1,0 were assigned to the 0/1 block. The rest of the

values of treatment combinations and their assigned block u/h were calculated using Excel and the results were displayed as in the table below.

Table 4.1. Confounding a Box-Benhken design in 3 blocks

Run	A	B	C	L1	L2	u	h	Yield
1	-1	-1	0	-2	-1	0	1	48.33
2	-1	1	0	0	-1	0	1	67.7
3	1	-1	0	0	1	0	1	66.33
4	1	1	0	2	1	0	1	63.65
5	-1	0	-1	-1	-2	1	0	64.09
6	-1	0	1	-1	0	1	0	55.96
7	1	0	-1	1	0	1	0	82.7
8	1	0	1	1	2	1	0	64.33
9	0	-1	-1	-1	-1	1	1	68
10	0	-1	1	-1	1	1	1	59.7
11	0	1	-1	1	-1	1	1	84
12	0	1	1	1	1	1	1	62
13	0	0	0	0	0	0	0	65.67
14	0	0	0	0	0	0	0	65.67
15	0	0	0	0	0	0	0	65.67

Thus the three blocks were generated as follows using the contrasts and distributing the stationary points equally among the blocks as shown below.

$$\begin{array}{ccc}
 0/1 & 1/0 & 1/1 \\
 \begin{bmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
 \end{array} \quad (4.1)$$

4.2 Computation of the estimates after blocking

The estimates were determined by getting the levels for each of the factors in the design. interaction components of the model. This was done in table below.

These included the liner, quadratic and

Table 4.2:Design for computing estimates after blocking

x1	x2	x3	X ² 1	X ² 2	X ² 3	X1X2	X1X3	X2X3	Y
-1	-1	0	1	1	0	1	0	0	48.33
-1	1	0	1	1	0	-1	0	0	67.7
1	-1	0	1	1	0	-1	0	0	66.33
1	1	0	1	1	0	1	0	0	63.65
0	0	0	0	0	0	0	0	0	65.67
-1	0	-1	1	0	1	0	1	0	64.09
-1	0	1	1	0	1	0	-1	0	55.96
1	0	-1	1	0	1	0	-1	0	82.7
1	0	1	1	0	1	0	1	0	64.33
0	0	0	0	0	0	0	0	0	65.67
0	-1	-1	0	1	1	0	0	1	68
0	-1	1	0	1	1	0	0	-1	59.7
0	1	-1	0	1	1	0	0	-1	84
0	1	1	0	1	1	0	0	1	62
0	0	0	0	0	0	0	0	0	65.67

From appendix II for design No.1

$$A = \frac{1}{8}, B = \frac{1}{4}, C_1 = \frac{1}{16}, D_1 = \frac{1}{4}, S = 2, \text{ and } \eta_c = 3$$

The grand total was obtained using (3.2) as

$$\{Oy\} = 983.8 \quad (4.2)$$

Then using (3.3) we obtain

$$\left. \begin{aligned} b_0 &= 65.67, \\ b_1 &= \frac{40.93}{8} = 5.11, b_2 = \frac{34.99}{8} = 4.37, b_3 = \frac{-56.8}{8} = -7.1 \\ b_{12} &= \frac{-22.05}{4} = -5.513, b_{13} = \frac{-10.24}{4} = -2.56 \end{aligned} \right\} (4.3a)$$

$$\left. \begin{aligned} b_{11} &= \frac{513.09}{4} - \frac{1573.58}{16} - \frac{65.67}{2} = -2.911 \\ b_{22} &= \frac{519.71}{4} - \frac{1573.58}{16} - \frac{65.67}{2} = -1.256 \\ b_{33} &= \frac{540.78}{4} - \frac{1573.58}{16} - \frac{65.67}{2} = 4.011 \end{aligned} \right\} (4.3b)$$

From Appendix I $C_2 = D_2 = 0$, then we have

$$\left. \begin{aligned} b_{12} &= \frac{-22.05}{4} = -5.513, b_{13} = \frac{-10.24}{4} = -2.56 \\ b_{23} &= \frac{-13.7}{4} = -3.425 \end{aligned} \right\} (4.4)$$

4.3 The Analysis of Variance

The analysis of variance table is obtained as

follows using (3.4) the

Correction due to the means:

$$\frac{983.8^2}{15} = 64524.16 \quad (4.5)$$

Using (3.5) Sum of squares due to linear

terms

$$= \frac{1}{8} (40.93^2 + 34.99^2 + (-56.8^2)) = 765.73(4.6)$$

And sum of squares due to second degree

terms:

Using (3.6) Due to interaction terms:

$$= \frac{1}{4} (-22.05^2 + -10.24^2 + -13.7^2) = 194.68(4.7)$$

And using (3.7) Due to quadratic terms

$$= 64628.85 - 64524.16 = 104.56 \quad (4.8)$$

And using (3.8) total sum of squares after

correction for the mean:

$$= 65612.5 - 64524.16$$

$$= 1088.333 \quad (4.9)$$

The above results were displayed in the

ANOVA table as shown below

Table 4.3 Analysis of Variance Table

	s.s	d.f	m.s
Due to linear	765.73	3	255.24
Due to interaction terms	194.68	3	32.45
Due to quadratic terms	104.56	3	34.85
Residuals	23.36	5	4.67
Total after eliminating the means	1088.33	14	

The observations recorded at the center points (65.67, 65.67, and 65.67) had no blocking of the design (that is they were made

entirely in random order), which means that these observations at the center points had (k-1) degrees of freedom i.e 3-1=2 for

estimating the error variance. The sum of squares for deviations from their mean is zero. Since any presence of the error sum of squares had only two degrees of freedom, it then led to a very insensitive test and

4.4 Elimination of Block Effects

The design experiment was carried out in three blocks of five observations, and then the elimination of blocks affects the residual sum of squares and due to second order terms. The block means were given as follows;

$$\left. \begin{aligned} Mb_1 &= \frac{311.68}{5} = 62.34, \quad Mb_2 = \frac{332.75}{5} = 66.55, \\ \text{and } Mb_3 &= \frac{339.37}{5} = 67.87 \end{aligned} \right\} (4.10)$$

Then the residual sum of squares was given as follows;

$$\frac{(311.68)^2 + (332.75)^2 + (339.37)^2}{5} - \frac{(983.8)^2}{15} = 83.63$$

The block effect is not same as the residual and this indicates presence of random effects within the experiment. On assumption that

provided no more than the indication that the remainder sum of squares was not of the right order of the magnitude hence the presence of random effects.

the model is adequate; the mean square of 4.67 from appendix V provides an estimate $\hat{\sigma}^2$ of σ^2 , which was used to calculate the standard errors of the variance coefficients which were calculated using (3.9) to obtain the variances and covariance of the various estimates as follows;

$$V(b_0) = \frac{4.67}{3} = 1.57$$

$$S.E(b_0) = \sqrt{1.57} = 1.25$$

$$V(b_i) = \frac{1}{8}(4.67) = .58$$

$$S.E(b_i) = \sqrt{.58} = .76$$

$$V(b_{ii}) = \left[\frac{1}{4} + \frac{1}{12}\right]4.67 = 1.56$$

$$S.E(b_{ii}) = \sqrt{1.56} = 1.29$$

$$V(b_{ij}) = \frac{1}{4}(4.67) = 1.17$$

$$Cov(b_0 b_{ii}) = -\frac{1}{12}(4.67) = -.390$$

$$S.E(b_{ij}) = \sqrt{1.17} = 1.08$$

$$Cov(b_{ii} b_{jj}) = \left[\frac{-1}{16} + \frac{1}{12} \right] 4.67 = 0.097$$

This yields the variance-covariance matrix below

$$\begin{bmatrix} 1.57 & 0 & 0 & 0 & -.39 & -.39 & -.39 & 0 & 0 & 0 \\ 0 & .58 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .58 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .58 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.39 & .097 & 0 & 0 & 1.56 & .097 & .097 & 0 & 0 & 0 \\ -.39 & .097 & 0 & 0 & 0 & 1.56 & 0 & 0 & 0 & 0 \\ -.39 & .097 & 0 & 0 & 0 & 0 & 1.56 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.17 \end{bmatrix}$$

Since the experimental error variance is different for the different blocks, it indicates that the error variance can change from one

block to another due to some extraneous sources of variation from the source of milk or origin of kefir grain over a period of time

5. Conclusion

The process of growing is affected by random factors and this should be controlled when growing kefir grains. From this study blocking was found to be effective in grouping the runs in to separate blocks to determine the presence of random factors during the growth of kefir grains .Hence the study confirm that there are random effects

apart from the known factors i.e milk fat content, speed of rotations and incubation time that might affect the growth of kefir grains using milk. These factors should be controlled or eliminated since they can interfere with the structure composition of probiotics that can are nutritious and good for health.

6. Recommendations

The 'study found out that there is presence of random factors during the growth of kefir grains using milk and hence its

recommended to control or eliminate them during the growth of kefir grain for good production of probiotics.

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APPENDIX 1

Design	A	B	C1	C2	D1	D2	S	Center Points N0	Redu- ndancy Factor	Non- Sphericity Index I
1	1/8	1/4	-1/16	0	1/4	0	2	3	1.2	0.38
2	1/12	1/8	-1/48	0	1/4	0	2	3	1.6	0
3	1/16	1/12	-1/96	0	1/4	0	2	6	1.9	0.17
4	1/24	17/216	-10/216	-1/216	1/16	1/8	3	6	1.7	.23
5	1/24	1/16	-1/144	0	1/8	0	3	6	1.6	0
6	1/40	1/30	-1/120	-1/720	1/16	1/8	3	10	2.2	.25
7	1/64	17/512	1/512	-7/512	1/16	1/32	4	10	2.4	.09
8	1/80	1/48	-1/600	0	1/32	0	5	12	2.3	.06
9	1/64	23/1024	-9/1024	-1/1024	1/32	1/16	4	12	2.1	.16
10	1/96	3072	-7/3072	-1/3072	1/32	1/16	4	12	2.5	.18

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